RICCI FLOW WITH SURGERY ON FOUR-MANIFOLDS WITH POSITIVE ISOTROPIC CURVATURE

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Abstract

In this paper we study the Ricci flow on compact four-manifolds with positive isotropic curvature and with no essential incompressible space form. We establish a long-time existence result of the Ricci flow with surgery on four-dimensional manifolds. As a consequence, we obtain a complete proof to the main theorem of Hamilton in [**21**]. During the proof we have actually provided, up to slight modifications, all necessary details for the part from Section 1 to Section 5 of Perelman's second paper [**32**] on the Ricci flow to approach the Poincaré conjecture.

CONTENTS

1. Introduction

Let M^n be a compact n-dimensional Riemannian manifold with metric $g_{ij}(x)$. The Ricci flow is the following evolution equation

$$(1.1) \qquad \frac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t), \text{ for } x \in M \text{ and } t > 0,$$

with $g_{ij}(x,0) = g_{ij}(x)$, where $R_{ij}(x,t)$ is the Ricci curvature tensor of the evolving metric $g_{ij}(x,t)$. This evolution system was initially introduced by Hamilton in [15]. Now it has been found to be a powerful tool to understand the geometry, topology and complex structure of manifolds (see for example [15], [16], [17], [21], [22], [5], [24], [2] [10], [9], [31], [32], [4], [3] etc.)

One of the main topics in modern mathematics is to understand the topology of compact three dimensional and four dimensional manifolds. The idea to approach this problem via the Ricci flow is to evolve the initial metric by the evolution equation (1.1), and try to study the geometries under the evolution. The key point of this method is to get the long-time behavior of the solutions of the Ricci flow. For a compact three (or four) dimensional Riemannian manifold with positive Ricci curvature (or a positive curvature operator, respectively) as initial data, Hamilton [15] (or [16] respectively) proved that the solution to the Ricci flow keeps shrinking and tends to a compact manifold with positive constant curvature before the solution vanishes. Consequently, a compact three-manifold with positive Ricci curvature or a compact four-manifold with positive curvature operator is diffeomorphic to the round sphere or a quotient of it by a finite group of fixed point free isometrics in the standard metric. In these classical cases, the singularities are formed everywhere simultaneously and with the same rates.

Note that even though the Ricci flow may develop singularities everywhere at the same time, the singularities can still be formed with different rates. The general case is that the Ricci flow may develop singularities in some parts while keeping smooth in other parts for general initial metrics. This suggests that we have to consider the structures of all the singularities (fast or slow forming). For the general case, naturally one would like to cut off the singularities and to continue the Ricci flow. If the Ricci flow still develops singularity after a while, one can do the surgeries and run the Ricci flow again. By repeating this procedure, one will get a kind of "weak" solution to the Ricci flow. Furthermore, if the "weak" solution has only a finite number of surgeries at any finite time interval and one can remember what had been cut during the surgeries, and the "weak" solution has a well-understood long-time behavior, then one will also get the topology structure of the initial manifold. This surgerically modified Ricci flow was initially developed by Hamilton [21] for compact four-manifolds. More recently, the idea of the Ricci flow with surgery was further developed by Perelman [32] for compact three-manifolds.

Let us give a brief description of the arguments of Hamilton in [21]. Recall that a Riemannian four-manifold is said to have **positive isotropic curvature** if for every orthonormal four-frame the curvature

tensor satisfies

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} > 2R_{1234}.$$

An **incompressible space form** N^3 in a four-manifold M^4 is a three-dimensional submanifold diffeomorphic to $\mathbb{S}^3/\Gamma$ (the quotient of the three-sphere by a group of isometries without fixed point) such that the fundamental group $\pi_1(N^3)$ injects into $\pi_1(M^4)$. The space form is said to be **essential** unless $\Gamma = \{1\}$, or $\Gamma = \mathbb{Z}_2$ and the normal bundle is nonorientable. In [**21**], Hamilton considered a compact four-manifold M^4 with no essential incompressible space-form and with a metric of positive isotropic curvature. He used this metric as initial data, and evolved it by the Ricci flow. From the evolution equations of curvatures, one can easily see that the curvature will become unbounded in finite time. Under the positive isotropic curvature assumption, he proved that as the time tends to the first singular time, either the solution has positive curvature operator everywhere, or it contains a neck, a region where the metric is very close to the product metric on $\mathbb{S}^3 \times \mathbb{I}$, where $\mathbb{I}$ is an interval and $\mathbb{S}^3$ is a round three-sphere, or a quotient of this by a finite group acting freely. When the solution has positive curvature operator everywhere, it is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$ by [**16**], so the topology of the manifold is understood and one can throw it away. When there is a neck in the solution, he used the no essential incompressible space form assumption to conclude that the neck must be $\mathbb{S}^3 \times \mathbb{I}$ or $\mathbb{S}^3 \times \mathbb{I}/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts antipodally on $\mathbb{S}^3$ and by reflection on $\mathbb{I}$. For the first case, one can replace $\mathbb{S}^3 \times \mathbb{I}$ with two caps (i.e. two copies of the differential four-ball $\mathbb{B}^4$) by cutting the neck and rounding off the neck. For the second case, one can do the quotient surgery to eliminate an $\mathbb{RP}^4$ summand. In [**21**], Hamilton performed these cutting and gluing surgery arguments so carefully that the positive isotropic curvature assumption and the improved pinching estimates are preserved under the surgeries. It is not hard to show that, after surgery, the new manifold still has no essential incompressible space form. Then by using this new manifold as initial data, one can run the Ricci flow and do the surgeries again. These arguments were given in Section A-D of [**21**]. In the last section (Section E) of [**21**], Hamilton showed that after a finite number of surgeries in finite time, and after discarding a finite number of pieces which are diffeomorphic to $\mathbb{S}^4$, $\mathbb{RP}^4$, the solution becomes extinct. This concludes that the four-manifold is diffeomorphic to $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{S}^1$, the twisted product $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$ (i.e., $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1 = \mathbb{S}^3 \times \mathbb{S}^1/\mathbb{Z}_2$, where $\mathbb{Z}_2$ flips $\mathbb{S}^3$ antipodally and rotates $\mathbb{S}^1$ by 180^0), or a connected sum of them.

The celebrated paper [**32**] tells us how to recognize the formation of singularities and how to perform the surgeries. One can see from Section A to D of [**21**] that every statement is accurate and every proof is complete, precise and detailed. Unfortunately, the last section (Section

E) contains some unjustified statements, which have been known for the experts in this field for several years. For example, one can see the comment of Perelman in [**32**] (Page 1, the second paragraph) and one can also check that the proof of Theorem E 3.3 of [**21**] is incomplete (in Proposition 3.4 of the present paper, we will prove a stronger version of Theorem E 3.3 of [**21**]). The key point is how to prevent the surgery times from accumulating (furthermore, it requires to perform only a finite number of surgeries in each finite time interval). By inspecting the last section of [**21**], it seems that surgeries were taken on the parts where the singularities are formed from the global maximum points of curvature. Intuitively, the other parts, where the curvatures go to infinity also but cannot be comparable to the global maximums, will still develop singularities shortly after surgery if one only performs the surgeries for the global maximum points of curvature. To prevent the surgery times from accumulating, one needs to cut off those singularities (not just the curvature maximum points) also. This means that one needs to perform surgeries for all singularities. Another problem is that, when one performs the surgeries with a given accuracy at each surgery time, it is possible that the errors may add up to a certain amount which causes the surgery times to accumulate. To prevent this from happening, as time goes on, successive surgeries must be performed with increasing accuracy.

Recently, Perelman [**31**], [**32**] presented the striking ideas of how to understand the structures of all singularities of the three-dimensional Ricci flow, how to find "fine" necks, how to glue "fine" caps, and how to use rescaling to prove that the times of surgery are discrete. When using rescaling arguments for surgically modified solutions of the Ricci flow, one encounters the difficulty of how to apply Hamilton's compactness theorem, which works only for smooth solutions. To overcome the difficulty, Perelman argued in [**32**] by choosing the cutoff radius in neck-like regions small enough to push the surgical regions far away in space. But we still have difficulty in taking a smooth limit since Shi's interior derivative estimate might not available, and so one cannot be certain that Hamilton's compactness result holds when only having the bound on curvatures. This is discussed in [**3**] and this paper.

In this paper, inspired by Perelman's works, we will study the Ricci flow on compact four-manifolds with positive isotropic curvature and with no essential incompressible space-form. We will give a complete proof for the main theorem of Hamilton in [**21**]. One of our major contributions in this paper is to establish several time-extension results for the surgical solutions in the proof of the discreteness of surgery times, so that the surgical solutions are smooth on some uniform (small) time intervals (on compact subsets) and Hamilton's compactness theorem is

still applicable. In Perelman's works [**31**, **32**], the universal noncollaps-
ing property of singularity models is a crucial fact to prove the surviving
of noncollapsing property under surgery. Another feature of this paper
is our proof of this crucial fact. In dimension three, one obtains this
by using Perelman's classification of three-dimensional shrinking Ricci
solitons with nonnegative curvature (see [**3**] for the details). But in the
present four dimension case, we are not able to obtain a complete classi-
fication for shrinking solitons. In the previous version, we presented an
argument to obtain the universal noncollapsing for shrinking solitons.
But, as pointed out to us by Joerg Enders, that argument contains a
gap. Fortunately in the present version, we find a new argument, with-
out appealing to a classification of shrinking Ricci solitons, to get the
universal noncollapsing for all possible singularity models.

During the proof we have actually provided, up to slight modifica-
tions, all necessary details for the part from Section 1 to Section 5 of
Perelman's second paper [**32**] on Ricci flow to approach the Poincaré
conjecture. The complete details of the arguments to both Poincaré and
Thurston's geometrization conjecture in three-dimension can be found
in the recent paper of H.-D. Cao and the second author in [**3**]. We
also refer the readers to the recent preprints of Kleiner-Lott [**26**] and
Morgan-Tian [**30**].

The main result of this paper is the following long-time existence
theorem.

Theorem 1.1.   *Let M^4 be a compact four-manifold with no essential
incompressible space-form and with a metric g_{ij} of positive isotropic
curvature. Then we have a finite collection of smooth solutions $g_{ij}^{(k)}(t)$,
$k = 0, 1, \ldots, m$, to the Ricci flow, defined on $M_k^4 \times [t_k, t_{k+1})$, $(0 = t_0 <
\cdots < t_{m+1})$ with $M_0^4 = M^4$ and $g_{ij}^{(0)}(t_0) = g_{ij}$, which go singular as
$t \to t_{k+1}$, such that the following properties hold:*

  (i) *for each $k = 0, 1, \ldots, m - 1$, the compact (possibly disconnected)
      four-manifold M_k^4 contains an open set Ω_k such that the solution
      $g_{ij}^{(k)}(t)$ can be smoothly extended to $t = t_{k+1}$ over Ω_k;*

 (ii) *for each $k = 0, 1, \ldots, m - 1$, $(\Omega_k, g_{ij}^{(k)}(t_{k+1}))$ and $(M_{k+1}^4,
      g_{ij}^{(k+1)}(t_{k+1}))$ contain compact (possible disconnected) four-dimen-
      sional submanifolds with smooth boundary, which are isometric
      and then can be denoted by N_k^4;*

(iii) *for each $k = 0, 1, \ldots, m-1$, $M_k^4 \setminus N_k^4$ consists of a finite number of
      disjoint pieces diffeomorphic to $\mathbb{S}^3 \times \mathbb{I}$, $\mathbb{B}^4$ or $\mathbb{RP}^4 \setminus \mathbb{B}^4$, while $M_{k+1}^4 \setminus
      N_k^4$ consists of a finite number of disjoint pieces diffeomophic to
      $\mathbb{B}^4$;*

(iv) *for $k = m$, M_m^4 is diffeomorphic to the disjoint union of a finite number of $\mathbb{S}^4$, or $\mathbb{RP}^4$, or $\mathbb{S}^3 \times \mathbb{S}^1$, or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$, or $\mathbb{RP}^4 \# \mathbb{RP}^4$.*

As a direct consequence we have the following classification result of Hamilton [21].

Corollary 1.2 (Hamilton [21]). *A compact four-manifold with no essential incompressible space-form and with a metric of positive isotropic curvature is diffeomorphic to $\mathbb{S}^4$, or $\mathbb{RP}^4$, or $\mathbb{S}^3 \times \mathbb{S}^1$, or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$, or a connected sum of them.*

In the recent preprint [25], Huisken and Sinestrari studied the mean curvature flow with surgeries of two-convex n-dimensional hypersurfaces of Euclidean space $\mathbb{R}^{n+1}$ (with $n \geq 3$). For four-dimensional hypersurfaces, the condition of two convexity is implied by nonnegative isotropic curvature. Thus the above corollary for the hypersurface case is also obtained in [25].

This paper contains five sections and an appendix. In Section 2 we recall the pinching estimates of Hamilton obtained in [21] and present two useful geometric properties for complete noncompact Riemannian manifolds with positive sectional curvature. The usual way to understand the singularities of the Ricci flow is to take a rescaling limit and to find the structure of the limiting models. In Section 3 we study the limiting models, so called ancient κ-solutions. We will establish the uniform κ-noncollapsing, compactness and canonical neighborhood structures for ancient κ-solutions. These generalize the analogs results of Perelman [31] from three-dimension to four-dimension. Section 4 extends the canonical neighborhood characterization to any solution of the Ricci flow with positive isotropic curvature, and describes the structure of the solution at the singular time. In Section 5, as Perelman in [32], we will define the Ricci flow with surgery, and by a long inductive argument, we will obtain a long-time existence result for the surgically modified Ricci flow so that the solution becomes extinct in finite time and takes only a finite number of surgeries. This will give the proof of the main theorem. In the appendix we will prove the curvature estimates for the standard solutions and give the canonical neighborhood description of the standard solution in dimension four, which are used in Section 5 for the surgery arguments.

2. Preliminaries

Consider a four-dimensional compact Riemannian manifold M^4. The curvature tensor of M^4 may be regarded as a symmetric bilinear form $M_{\alpha\beta}$ on the space of real forms Λ^2. It is well known that one can decompose Λ^2 into $\Lambda_+^2 \oplus \Lambda_-^2$ as eigen-spaces of the Hodge star operator with eigenvalues ± 1. This gives a block decomposition of the curvature operator $(M_{\alpha\beta})$ as

$$(M_{\alpha\beta}) = \begin{pmatrix} A & B \\ {}^t B & C \end{pmatrix}.$$

It was shown in Lemma A2.1 of [21] that a four-manifold has positive isotropic curvature if and only if

$$a_1 + a_2 > 0 \text{ and } c_1 + c_2 > 0$$

where $a_3 \geq a_2 \geq a_1$, $c_3 \geq c_2 \geq c_1$ are eigenvalues of the matrices A and C respectively.

Let $\{X_1, X_2, X_3, X_4\}$ be a positive oriented orthonormal basis of one-forms. Then $\varphi_1 = X_1 \wedge X_2 + X_3 \wedge X_4$, $\varphi_2 = X_1 \wedge X_3 + X_4 \wedge X_2$, $\varphi_3 = X_1 \wedge X_4 + X_2 \wedge X_3$ is a basis of Λ_+^2 and $\psi_1 = X_1 \wedge X_2 - X_3 \wedge X_4$, $\psi_2 = X_1 \wedge X_3 - X_4 \wedge X_2$, $\psi_3 = X_1 \wedge X_4 - X_2 \wedge X_3$ is a basis of Λ_-^2. It is easy to check $trA = trC = \frac{1}{2}R$ by using this orthonormal basis and the Bianchi identity.

Since B may not be symmetric, its eigenvalues need to be explained as follows. For an appropriate choice of orthonormal bases y_1^+, y_2^+, y_3^+ of Λ_+^2 and y_1^-, y_2^-, y_3^- of Λ_-^2 the matrix

$$B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix}$$

with $0 \leq b_1 \leq b_2 \leq b_3$. They are actually the eigenvalues of the symmetric matrices $\sqrt{B^t B}$ or $\sqrt{{}^t BB}$.

In [21] Hamilton proved that the Ricci flow on a compact four-manifold preserves positive isotropic curvature, and obtained the following improving pinching estimate.

Lemma 2.1 (Theorem B1.1 and Theorem B2.3 of [21]). *Given an initial metric on a compact four-manifold with positive isotropic curvature, there exist positive constants ρ, $\Lambda, P < +\infty$ depending only on the initial metric, such that the solution to the Ricci flow satisfies*

$$(2.1) \qquad a_1 + \rho > 0 \text{ and } c_1 + \rho > 0,$$

$$(2.2) \quad \max\{a_3, b_3, c_3\} \leq \Lambda(a_1 + \rho) \text{ and } \max\{a_3, b_3, c_3\} \leq \Lambda(c_1 + \rho),$$

and

$$(2.3) \qquad \frac{b_3}{\sqrt{(a_1 + \rho)(c_1 + \rho)}} \leq 1 + \frac{\Lambda e^{Pt}}{\max\{\log \sqrt{(a_1 + \rho)(c_1 + \rho)}, 2\}}$$

at all points and all times.

This lemma tells us that as we consider the Ricci flow for a compact four-manifold with positive isotropic curvature, any rescaling limit along a sequence of points where the curvatures become unbounded must still have positive isotropic curvature and satisfy the following restricted isotropic curvature pinching condition

$$(2.4) \qquad\qquad a_3 \leq \Lambda a_1, \quad c_3 \leq \Lambda c_1, \quad b_3^2 \leq a_1 c_1.$$

In the rest of this section, we will give two useful geometric properties for Riemannian manifolds with nonnegative sectional curvature.

Let (M^n, g_{ij}) be an n-dimensional complete Riemannian manifold and let ε be a positive constant. We call an open subset $N \subset M^n$ to be an ε-**neck of radius** r if $(N, r^{-2}g_{ij})$ is ε-close, in $C^{[\varepsilon^{-1}]}$ topology, to a standard neck $\mathbb{S}^{n-1} \times \mathbb{I}$ with $\mathbb{I}$ of the length $2\varepsilon^{-1}$ and $\mathbb{S}^{n-1}$ of the scalar curvature 1.

Proposition 2.2. *There exists a constant $\varepsilon_0 = \varepsilon_0(n) > 0$ such that every complete noncompact Riemannian manifold (M^n, g_{ij}) of nonnegative sectional curvature has a positive constant r_0 such that any ε-neck of radius r on (M^n, g_{ij}) with $\varepsilon \leq \varepsilon_0$ must have $r \geq r_0$.*

Proof. We argue by contradiction. Suppose there exists a sequence of positive constants $\varepsilon^\alpha \to 0$ and a sequence of n-dimensional complete noncompact Riemannian manifolds $(M^\alpha, g_{ij}^\alpha)$ with nonnegative sectional curvature such that for each fixed α, there exists a sequence of ε^α-necks N_k of radius at most $1/k$ on M^α with centers P_k divergent to infinity.

Fix a point P on the manifold M^α and connect each P_k to P by a minimizing geodesic γ_k. By passing to subsequence we may assume the angle θ_{kl} between geodesic γ_k and γ_l at P is very small and tends to zero as $k, l \to +\infty$, and the length of γ_{k+1} is much bigger than the length of γ_k. Let us connect P_k to P_l by a minimizing geodesic η_{kl}. For each fixed $l > k$, let $\tilde{P}_k$ be a point on the geodesic γ_l such that the geodesic segment from P to $\tilde{P}_k$ has the same length as γ_k and consider the triangle $\Delta PP_k\tilde{P}_k$ in M^α with vertices P, P_k and $\tilde{P}_k$. By comparing with the corresponding triangle in the Euclidean plane $\mathbb{R}^2$ whose sides have the same corresponding lengths, Toponogov comparison theorem implies

$$d(P_k, \tilde{P}_k) \leq 2\sin\left(\frac{1}{2}\theta_{kl}\right) \cdot d(P_k, P).$$

Since θ_{kl} is very small, the distance from P_k to the geodesic γ_l can be realized by a geodesic ζ_{kl} which connects P_k to a point P_k' on the interior of the geodesic γ_l and has length at most $2\sin(\frac{1}{2}\theta_{kl}) \cdot d(P_k, P)$. Clearly the angle between ζ_{kl} and γ_l at the intersection point P_k' is $\frac{\pi}{2}$. Consider α to be fixed and sufficiently large. We claim that as k gets

large enough, each minimizing geodesic γ_l with $l > k$, connecting P to P_l, goes through the neck N_k.

Suppose not; then the angle between γ_k and ζ_{kl} at P_k is close to either zero or π, since P_k is in the center of an ε^α-neck and α is sufficiently large. If the angle between γ_k and ζ_{kl} at P_k is close to zero, we consider the triangle $\Delta PP_kP'_k$ in M^α with vertices P, P_k, and P'_k. By applying Toponogov comparison theorem to compare the angles of this triangle with those of the corresponding triangle in the Euclidean plane with the same corresponding lengths, we find that it is impossible. Thus the angle between γ_k and ζ_{kl} at P_k is close to π. We now consider the triangle $\Delta P_kP'_kP_l$ in M^α with the three sides ζ_{kl}, η_{kl} and the geodesic segment from P'_k to P_l on γ_l. We have seen that the angle of $\Delta P_kP'_kP_l$ at P_k is close to zero and the angle at P'_k is $\frac{\pi}{2}$. By comparing with corresponding triangle $\bar{\Delta}\bar{P}_k\bar{P}'_k\bar{P}_l$ in the Euclidean plane $\mathbb{R}^2$ whose sides have the same corresponding lengths, Toponogov comparison theorem implies

$$\angle\bar{P}_l\bar{P}_k\bar{P}'_k + \angle\bar{P}_l\bar{P}'_k\bar{P}_k \le \angle P_lP_kP'_k + \angle P_lP'_kP_k < \frac{3}{4}\pi.$$

This is impossible, since the length between $\bar{P}_k$ and $\bar{P}'_k$ is much smaller than the length from $\bar{P}_l$ to either $\bar{P}_k$ or $\bar{P}'_k$. So we have proved each γ_l with $l > k$ passes through the neck N_k.

Hence by taking a limit, we get a geodesic ray γ emanating from P which passes through all the necks N_k, $k = 1, 2, \ldots$, except a finite number of them. Throwing this finite number of necks, we may assume γ passes through all necks N_k, $k = 1, 2, \ldots$. Denote the center sphere of N_k by S_k, and their intersection points with γ by $p_k \in S_k \cap \gamma$, for $k = 1, 2, \ldots$.

Take sequence points $\gamma(m)$ with $m = 1, 2, \ldots$. For each fixed neck N_k, arbitrarily choose a point $q_k \in N_k$ near the center sphere S_k, and draw a geodesic segment γ^{km} from q_k to $\gamma(m)$. Now we claim that for any fixed neck N_l with $l > k$, γ^{km} will pass through N_l for all sufficiently large m.

We argue by contradiction. Let us place all necks N_i horizontally so that the geodesic γ passes through each N_i from the left to the right. We observe that the geodesic segment γ^{km} must pass through the right half of N_k; otherwise γ^{km} can not be minimal. Then as m is large enough, the distance from p_l to the geodesic segment γ^{km} must be achieved by the distance from p_l to some interior point p_k' of γ^{km}. Let us draw a minimal geodesic η from p_l to the interior point p_k' with the angle at the intersection point $p_k' \in \eta \cap \gamma^{km}$ is $\frac{\pi}{2}$. Suppose the claim is false. Then the angle between η and γ at p_l is close to 0 or π since ε^α is small.

If the angle between η and γ at p_l is close to 0, we consider the triangle $\Delta p_l p_k' \gamma(m)$ and construct a comparison triangle $\bar{\Delta}\bar{p}_l\bar{p}_k'\bar{\gamma}(m)$ in the plane with the same corresponding length. Then by Toponogov

comparison, we see that the sum of the inner angles of the comparison triangle $\bar{\Delta}\bar{p}_l\bar{p}_k{}'\bar{\gamma}(m)$ is less than $3\pi/4$, which is impossible.

If the angle between η and γ at p_l is close to π, by drawing a minimal geodesic from ξ from q_k to p_l, we see that ξ must pass through the right half of N_k and the left half of N_l; otherwise ξ can not be minimal. Thus the three inner angles of the triangle $\Delta p_l p_k{}' q_k$ are almost $0, \pi/2, 0$ respectively. This is also impossible by Toponogov comparison theorem.

Hence we have proved that the geodesic segment γ^{km} passes through N_l as m large enough.

Consider the triangle $\Delta p_k q_k \gamma(m)$ with two long sides $\overline{p_k\gamma(m)}(\subset \gamma)$ and $\overline{q_k\gamma(m)}(=\gamma^{km})$. For any $s > 0$, choose two points $\tilde{p}_k$ on $\overline{p_k\gamma(m)}$ and $\tilde{q}_k$ on $\overline{q_k\gamma(m)}$ with $d(p_k, \tilde{p}_k) = d(q_k, \tilde{q}_k) = s$. By Toponogov comparison theorem, we have

$$
\left(\frac{d(\tilde{p}_k, \tilde{q}_k)}{d(p_k, q_k)}\right)^2
$$

$$
= \frac{d(\tilde{p}_k, \gamma(m))^2 + d(\tilde{q}_k, \gamma(m))^2 - 2d(\tilde{p}_k, \gamma(m))d(\tilde{q}_k, \gamma(m))\cos\bar{\measuredangle}(\tilde{p}_k\gamma(m)\tilde{q}_k)}{d(p_k, \gamma(m))^2 + d(q_k, \gamma(m))^2 - 2d(p_k, \gamma(m))d(q_k, \gamma(m))\cos\bar{\measuredangle}(p_k\gamma(m)q_k)}
$$

$$
\geq \frac{d(\tilde{p}_k, \gamma(m))^2 + d(\tilde{q}_k, \gamma(m))^2 - 2d(\tilde{p}_k, \gamma(m))d(\tilde{q}_k, \gamma(m))\cos\bar{\measuredangle}(\tilde{p}_k\gamma(m)\tilde{q}_k)}{d(p_k, \gamma(m))^2 + d(q_k, \gamma(m))^2 - 2d(p_k, \gamma(m))d(q_k, \gamma(m))\cos\bar{\measuredangle}(\tilde{p}_k\gamma(m)\tilde{q}_k)}
$$

$$
= \frac{(d(\tilde{p}_k, \gamma(m)) - d(\tilde{q}_k, \gamma(m)))^2 + 2d(\tilde{p}_k, \gamma(m))d(\tilde{q}_k, \gamma(m))(1 - \cos\bar{\measuredangle}(\tilde{p}_k\gamma(m)\tilde{q}_k))}{(d(\tilde{p}_k, \gamma(m)) - d(\tilde{q}_k, \gamma(m)))^2 + 2d(p_k, \gamma(m))d(q_k, \gamma(m))(1 - \cos\bar{\measuredangle}(\tilde{p}_k\gamma(m)\tilde{q}_k))}
$$

$$
\geq \frac{d(\tilde{p}_k, \gamma(m))d(\tilde{q}_k, \gamma(m))}{d(p_k, \gamma(m))d(q_k, \gamma(m))}
$$

$$
\to 1
$$

as $m \to \infty$, where $\bar{\measuredangle}(p_k\gamma(m)q_k)$ and $\bar{\measuredangle}(\tilde{p}_k\gamma(m)\tilde{q}_k)$ are the the corresponding angles in the corresponding comparison triangles.

Letting $m \to \infty$, we see that γ^{km} has a convergent subsequence whose limit γ^k is a geodesic ray passing through all N_l with $l > k$. Denote by $p_j = \gamma(t_j), j = 1, 2, \ldots$. From the above computation, we deduce that

$$
d(p_k, q_k) \leq d(\gamma(t_k + s), \gamma^k(s))
$$

for all $s > 0$.

Let $\varphi(x) = \lim_{t \to +\infty}(t - d(x, \gamma(t)))$ be the Busemann function constructed from the ray γ. Note that the level set $\varphi^{-1}(\varphi(p_j)) \cap N_j$ is close to the center sphere S_j for any $j = 1, 2, \ldots$. Now let q_k be any fixed point in $\varphi^{-1}(\varphi(p_k)) \cap N_k$. By the definition of Busemann function φ associated to the ray γ, we see that $\varphi(\gamma^k(s_1)) - \varphi(\gamma^k(s_2)) = s_1 - s_2$ for any $s_1, s_2 \geq 0$. Consequently, for each $l > k$, by choosing $s = t_l - t_k$, we see $\gamma^k(t_l - t_k) \in \varphi^{-1}(\varphi(p_l)) \cap N_l$. Since $\gamma(t_k + t_l - t_k) = p_l$, it follows

that

$$d(p_k, q_k) \le d(p_l, \gamma^k(s))$$

with $s = t_l - t_k > 0$. This implies that the diameter of $\varphi^{-1}(\varphi(p_k)) \cap N_k$ is not greater than the diameter of $\varphi^{-1}(\varphi(p_l)) \cap N_l$ for any $l > k$, which is a contradiction as l is much larger than k.

Therefore we have proved the proposition.   q.e.d.

In [20], Hamilton discovered an interesting result, called the finite bump theorem, about the influence of a bump of strictly positive curvature in a complete noncompact Riemannian manifold with nonnegative sectional curvature. Namely, minimal geodesic paths that go past the bump have to avoid it. The following result is in the same spirit as Hamilton's finite bump theorem.

Proposition 2.3. *Suppose (M^n, g) is a complete n-dimensional Riemanian manifold with nonnegative sectional curvature. Let $P \in M^n$ be fixed, and $P_j \in M^n$ a sequence of points and R_j a sequence of positive numbers with $d(P, P_j) \to +\infty$ and $R_j d(P, P_j)^2 \to +\infty$. If the sequence of marked manifolds $(M^n, R_j g, P_j)$ converges in C^∞_{loc} topology (in Cheeger sense) to a smooth manifold $(\tilde{M}^n, \tilde{g})$, then the limit $(\tilde{M}^n, \tilde{g})$ splits as the metric product of the form $\mathbb{R} \times N$, where N is a nonnegatively curved manifold of dimension $n - 1$.*

Proof. Let us denote by $|OQ| = d(O, Q)$ for the distance of two points $O, Q \in M^n$. Without loss of generality, we may assume that for each j,

(2.5) $$1 + 2|PP_j| \le |PP_{j+1}|.$$

Draw a minimal geodesic γ_j from P to P_j and a minimal geodesic σ_j from P_j to P_{j+1}, both parameterized by the arclength. By the compactness of unit sphere of the tangent space at P, $\{\gamma_j'(0)\}$ has a convergent subsequence. We may further assume

(2.6) $$\theta_j = |\angle(\gamma_j'(0), \gamma_{j+1}'(0))| < \frac{1}{j}.$$

Since $(M^n, R_j g, P_j)$ converges in C^∞_{loc} topology (in Cheeger sense) to a smooth marked manifold $(\tilde{M}^n, \tilde{g}, \tilde{P})$, by further choices of subsequences, we may also assume γ_j and σ_j converge to two geodesic rays $\tilde{\gamma}$ and $\tilde{\sigma}$ starting at $\tilde{P}$. We claim that $\tilde{\gamma} \cup \tilde{\sigma}$ forms a line in $\tilde{M}^n$. Since the sectional curvature of $\tilde{M}^n$ is nonnegative, then by Toponogov splitting theorem [7] the limit $\tilde{M}^n$ must split as $\mathbb{R} \times N$ isometrically.

To prove the claim, we argue by contradiction. Suppose $\tilde{\gamma} \cup \tilde{\sigma}$ is not a line; then for each j, there exist two points $A_j \in \gamma_j$ and $B_j \in \sigma_j$ such

that as $j \to +\infty$,

(2.7)
$$\begin{cases} R_j d(P_j, A_j) \to A > 0, \\ R_j d(P_j, B_j) \to B > 0, \\ R_j d(A_j, B_j) \to C > 0, \\ \text{but } A + B > C. \end{cases}$$
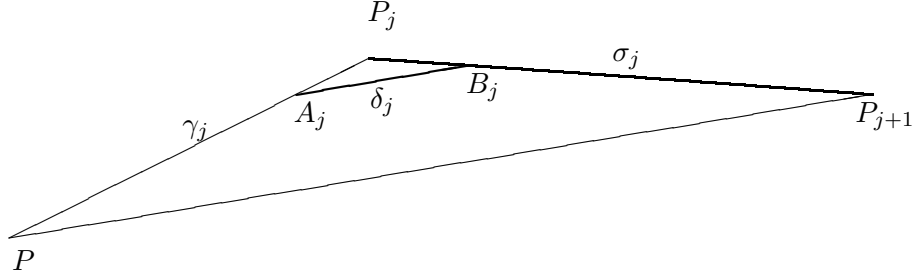
Now draw a minimal geodesic δ_j from A_j to B_j. Consider comparison triangle $\bar{\triangle} \bar{P}_j \bar{P} \bar{P}_{j+1}$ and $\bar{\triangle} \bar{P}_j \bar{A}_j \bar{B}_j$ in $\mathbb{R}^2$ with

$$|\bar{P}_j \bar{P}| = |P_j P|, |\bar{P}_j \bar{P}_{j+1}| = |P_j P_{j+1}|, |\bar{P} \bar{P}_{j+1}| = |P P_{j+1}|,$$

$$\text{and } |\bar{P}_j \bar{A}_j| = |P_j A_j|, |\bar{P}_j \bar{B}_j| = |P_j B_j|, |\bar{A}_j \bar{B}_j| = |A_j B_j|.$$

By Toponogov comparison theorem [**7**], we have

(2.8)
$$\angle \bar{A}_j \bar{P}_j \bar{B}_j \geq \angle \bar{P} \bar{P}_j \bar{P}_{j+1}.$$

On the other hand, by (2.6) and using the Toponogov comparison theorem again, we have

(2.9)
$$\angle \bar{P}_j \bar{P} \bar{P}_{j+1} \leq \angle P_j P P_{j+1} < \frac{1}{j},$$

and since $|\bar{P}_j \bar{P}_{j+1}| > |\bar{P} \bar{P}_j|$ by (2.5), we further have

(2.10)
$$\angle \bar{P}_j \bar{P}_{j+1} \bar{P} \leq \angle \bar{P}_j \bar{P} \bar{P}_{j+1} < \frac{1}{j}.$$

Thus the above inequalities (2.8)-(2.10) imply that

$$\angle \bar{A}_j \bar{P}_j \bar{B}_j > \pi - \frac{2}{j}.$$

Hence

(2.11) $$|\bar{A}_j \bar{B}_j|^2 \geq |\bar{A}_j \bar{P}_j|^2 + |\bar{P}_j \bar{B}_j|^2 - 2|\bar{A}_j \bar{P}_j| \cdot |\bar{P}_j \bar{B}_j| \cos\left(\pi - \frac{2}{j}\right).$$

Multiplying the above inequality by R_j and letting $j \to +\infty$, we get

$$C \geq A + B$$

which contradicts with (2.7). Therefore we have proved the proposition.

q.e.d.

Although the following result is not used in this paper, we think that it is still worthy of being included here.

Corollary 2.4. *Suppose (X, d) is a complete n-dimensional Alexandrov space with nonnegative curvature. Let $P \in X$ be fixed, $P_j \in X$ a sequence of points, and R_j a sequence of positive numbers with $d(P, P_j) \to +\infty$ and $R_j d^2(P, P_j) \to +\infty$. Then the marked spaces $(X, R_j^{\frac{1}{2}} d, P_j)$ have a (Gromov-Hausdorff) convergent subsequence such that the limit splits as the metric product of the form $\mathbb{R} \times N$, where N is a nonnegatively curved Alexandrov space.*

Proof. By the compactness theorem of Alexandrov spaces (see [**1**]), there is a subsequence of $(X, R_j^{\frac{1}{2}} d, P_j)$ which converges (in the sense of Gromov-Hausdorff) to a nonnegatively curved Alexandrov space $(\widetilde{X}, \widetilde{d}, \widetilde{P})$ of dimension $\leq n$. By Toponogov splitting theorem [**28**] for Alexandrov spaces, we only need to show that the limit $\widetilde{X}$ contains a line. Note that the same inequality (2.6) now follows from the compactness of the space of directions at a fixed point [**1**]. Since the Toponogov triangle comparison theorem still holds on Alexsandrov spaces (in fact, the notion of the curvature of general metric spaces is defined by Toponogov triangle comparison), the same argument of Proposition 2.3 proves the corollary.   q.e.d.

3. Ancient Solutions

A solution to the Ricci flow on a compact four-manifold with positive isotropic curvature develops singularities in finite time. The usual way to understand the formations of the singularities is to rescale the solution along the singularities and to try to take a limit for the rescaled sequences. According to Lemma 2.1, a rescaled limit will be a complete non-flat solution to the Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2 R_{ij}$$

on an ancient time interval $-\infty < t \leq 0$, called an **ancient solution**, which has positive isotropic curvature and satisfies the restricted isotropic curvature pinching condition (2.4). We remark that as we consider the general singularities (not necessarily those points coming from the maximum of the curvature), we don't know whether at a priori, the rescaled limit exists, and even assuming the existence, whether the limit has bounded curvature for each t. Nevertheless, in this section we will take the attention to those rescaled limits with bounded curvature.

According to Perelman [**31**], a solution to the Ricci flow is κ-**noncollapsed for scale** $r_0 > 0$ if we have the following statement: whenever we have

$$|Rm|(x, t) \leq r_0^{-2},$$

for all $t \in [t_0 - r_0^2, t_0]$, $x \in B_t(x_0, r_0)$, for some (x_0, t_0), there holds

$$\mathrm{Vol}_{t_0}(B_{t_0}(x_0, r_0)) \geq \kappa r_0^4.$$

Here we denote by $B_t(x_0, r_0)$ and Vol_{t_0} the geodesic ball centered at x_0 of radius r_0 with respect to the metric $g_{ij}(t)$ and the volume with respect to the metric $g_{ij}(t_0)$ respectively. It was shown by Perelman [**31**] that any rescaled limit obtained by blowing up a smooth solution to the Ricci flow on a compact manifold in finite time is κ-noncollapsed on all scales for some $\kappa > 0$.

We say a solution to the Ricci flow on a four-manifold is an **ancient κ -solution with restricted isotropic curvature pinching** (for some $\kappa > 0$) if it is a smooth solution to the Ricci flow on the ancient time interval $t \in (-\infty, 0]$ which is complete, has positive isotropic curvature and bounded curvature, and satisfies the restricted isotropic curvature pinching condition (2.4), as well as is κ-noncollapsed on all scales.

3.1. Splitting lemmas. To understand the structures of the solutions to the Ricci flow on a compact four-manifold with positive isotropic curvature, we are naturally led to investigate the ancient solutions which have positive isotropic curvature, satisfy the restricted isotropic curvature pinching condition (2.4) and are κ-noncollapsed for all scales. Note that the restricted isotropic curvature pinching condition (2.4) implies the curvature operator is nonnegative. In this subsection we will derive two useful splitting results **without** assuming the bounded curvature condition.

Lemma 3.1. *Let (M^4, g_{ij}) be a complete noncompact Riemannian manifold which satisfies the restricted isotropic curvature pinching condition (2.4) and has positive curvature operator. And let P be a fixed point in M^4, $\{P_l\}_{1 \leq l < +\infty}$ a sequence of points in M^4 and $\{R_l\}_{1 \leq l < +\infty}$ a sequence of positive numbers with $d(P, P_l) \to +\infty$ and $R_l d^2(P, P_l) \to +\infty$ as $l \to +\infty$, where $d(P, P_l)$ is the distance between P and P_l. Suppose $(M^4, R_l g_{ij}, P_l)$ converges in C^∞_{loc} topology to a smooth nonflat limit Y. Then Y must be isometric to $\mathbb{R} \times \mathbb{S}^3$ with the standard metric (up to a constant factor).*

Proof. By Proposition 2.3, we see that $Y = \mathbb{R} \times X$ for some smooth three-dimensional manifold X. Thus the block decomposition of the curvature operator has the form

$$(M_{\alpha\beta}) = \begin{pmatrix} A & A \\ A & A \end{pmatrix}.$$

The assumption that $b_3^2 \leq a_1 c_1$ in (2.4) implies that the matrix $A(= B = C)$ is a multiple of the identity. Since this is true at every point, it follows from the contracted second Bianchi identity that X has (positive) constant curvature, i.e., $X = \mathbb{S}^3/\Gamma$ for some group Γ of isometries without fixed points. It remains to show $X = \mathbb{S}^3$ (i.e., $\Gamma = \{1\}$).

Note that the original manifold M^4 is diffeomorphic to $\mathbb{R}^4$ by the positive curvature operator assumption. To show $X = \mathbb{S}^3$ (i.e., $\Gamma = \{1\}$) we only need to prove that $X = \mathbb{S}^3/\Gamma$ is incompressible in M^4. In the following we adapt Hamilton's argument in Theorem C 4.1 of [**21**] to noncompact manifolds.

Suppose $X = \mathbb{S}^3/\Gamma$ is not incompressible in M^4; then for j large, there exists a simply closed curve $\gamma \subset \mathbb{S}^3/\Gamma$ (the center space form passing through P_j) which is homotopically nontrivial in $\mathbb{S}^3/\Gamma$ but bounds a disk D^2 in M^4. By Lemma C4.2 in [**21**], we may assume the disk D^2 meets the center space form $\mathbb{S}^3/\Gamma$ only at γ, where it is transversal. Now construct a new manifold $\widehat{M}^4$ in the following way. As in [**21**] we can deform the metric in the neck around P_j a little so it is standard in a smaller neck but still has positive isotropic curvature everywhere. Since M^4 is simply connected, the connected and closed submanifold $\mathbb{S}^3/\Gamma$ of codimension 1 separates M^4 (see for example [**23**]). Cut M^4 open along the center space form $\mathbb{S}^3/\Gamma$ to get a (maybe disconnected) manifold with two boundary components $\mathbb{S}^3/\Gamma$, and double across the boundary to get $\widehat{M}^4$. The new manifold $\widehat{M}^4$ also has a metric of positive isotropic curvature since the boundary is flat extrinsically and we can double the metric. If $\widehat{M}^4$ contains a compact connected component and the above disk D^2 also lies in the compact connected component, then the same argument as in the proof of Theorem C4.1 of [**21**] derives a contradiction. Thus we may assume that the unique noncompact connected component of $\widehat{M}^4$, denoted by $\widehat{M}_1^4$, contains the disk D^2. Since M^4 has positive curvature operator, we know from [**8**] that there is a strictly convex exhausting function φ on M^4. We can define a function $\widehat{\varphi}$ on $\widehat{M}^4$ so that $\widehat{\varphi} = \varphi$ on each copy of M^4. Then as $c > 0$ is sufficiently large, the level set $\widehat{\varphi}^{-1}(c)$ is contained in the unique noncompact connected component $\widehat{M}_1^4$ and is strictly convex (in the sense that its second fundamental is strictly positive). Take our disk D^2 bounding the curve γ and perpendicular to the boundary in a neighborhood of the boundary, and double it across the boundary to get a sphere $\mathbb{S}^2$ which is $\mathbb{Z}_2$ invariant and intersects the boundary component transversally in γ. The homotopy class $[\gamma]$ is nontrivial in $\mathbb{S}^3/\Gamma$. Clearly the above two-sphere $\mathbb{S}^2$ is contained in the set $\{x \in \widehat{M} | \widehat{\varphi}(x) \leq c\}$ as $c > 0$ large enough, since φ is an exhausting on M^4. Now fix such a large positive constant c. Among all spheres which are $\mathbb{Z}_2$ invariant, contained in the manifold $\{x \in \widehat{M}^4 | \widehat{\varphi}(x) \leq c\}$, and intersect the $\mathbb{S}^3/\Gamma$ in the homotopy class $[\gamma] \neq 0$, there will be one of least area since the boundary of the manifold $\{x \in \widehat{M}^4 | \widehat{\varphi}(x) \leq c\}$ is strictly convex. This sphere must even have least area among all nearby spheres. For if a nearby sphere of less area divides in two parts bounding $[\gamma]$ in $\mathbb{S}^3/\Gamma$, one side or the other has less than half the area of the original sphere. We could then double

this half to get a sphere of less area which is $\mathbb{Z}_2$ invariant, contradicting the assumption that ours was of least area among this class. But the hypothesis of positive isotropic curvature implies there are no stable minimal two-spheres as was shown in [**27**]. Hence we get a contradiction unless $X = \mathbb{S}^3/\Gamma$ is incompressible in M^4.

Therefore we have proved the lemma.   q.e.d.

Lemma 3.2. *Let* $(M^4, g_{ij}(t))$ *be an ancient solution which has positive isotropic curvature and satisfies the restricted isotropic curvature pinching condition* (2.4). *If its curvature operator has a nontrivial null eigenvector somewhere at some time, then the solution is, up to a scaling, the evolving round cylinder* $\mathbb{R} \times \mathbb{S}^3$ *or a metric quotient of the round cylinder* $\mathbb{R} \times \mathbb{S}^3$.

Proof. Recall that the solution $g_{ij}(t)$ has nonnegative curvature operator everywhere and every time. Because the curvature operator of the ancient solution $g_{ij}(t)$ has a nontivial null eigenvector somewhere at some time, it follows from [**16**] (by using Hamilton's strong maximum principle) that at any earlier time the solution has null eigenvector everywhere and the Lie algebra of the holonomy group is restricted a proper subalgebra of $so(4)$. Since the ancient solution is non-flat and has positive isotropic curvature, we rule out the subalgebras $\{1\}$, $u(2)$, $so(2) \times so(2)$, $so(2) \times \{1\}$ as on $\mathbb{R}^4, \mathbb{CP}^2, \mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2 \times \mathbb{R}^2$ or a metric quotient of them. The only remaining possibility for the Lie subalgebra of the holonomy is $so(3)$.

Now the only way we get holonomy $so(3)$ is when in some basis we have $A = B = C$ in the curvature operator matrix, so that

$$(M_{\alpha\beta}) = \begin{pmatrix} A & A \\ A & A \end{pmatrix},$$

which corresponds to the fact that the metric $g_{ij}(t)$ is locally a product of $\mathbb{R} \times X$ for some smooth three-dimensional manifold X with curvature operator A. Then the inequality $b_3^2 \leq a_1 c_1$ in the restricted isotropic curvature pinching condition (2.4) implies that A is a multiple of the identity. Moreover, if this is true at every point, it follows from the contracted second Bianchi identity that the factor X has (positive) constant curvature. Consequently, X is compact and so for each t, the metric $g_{ij}(t)$ is isometric (up to a scaling) to the evolving metric of the round cylinder $\mathbb{R} \times \mathbb{S}^3$ or a metric quotient of it.   q.e.d.

3.2. Elliptic type estimate, canonical neighborhood decomposition for noncompact κ-solutions. The following elliptic type Harnack property for four-dimensional ancient κ-solutions with restricted isotropic curvature pinching will be crucial for the analysis of the structure of singularities of the Ricci flow on four-manifold with positive isotropic curvature. The analogous result for three-dimensional

ancient κ-solutions was implicitely given by Perelman in Section 11.7 of [**31**] and Section 1.5 of [**32**].

Proposition 3.3. *For any $\kappa > 0$, there exist a positive constant η and a positive function $\omega : [0, +\infty) \rightarrow (0, +\infty)$ with the following properties. Suppose we have a four-dimensional ancient κ-solution $(M^4, g_{ij}(t)), -\infty < t \leq 0$, with restricted isotropic curvature pinching. Then*

(i) *for every $x, y \in M^4$ and $t \in (-\infty, 0]$, there holds*

$$R(x, t) \leq R(y, t) \cdot \omega(R(y, t)d_t^2(x, y));$$

(ii) *for all $x \in M^4$ and $t \in (-\infty, 0]$, there hold*

$$|\nabla R|(x, t) \leq \eta R^{\frac{3}{2}}(x, t) \quad and \quad \left|\frac{\partial R}{\partial t}\right|(x, t) \leq \eta R^2(x, t).$$

Proof. Obviously we may assume the ancient κ-solution is not a metric quotient of the round neck $\mathbb{R} \times \mathbb{S}^3$.

(i) We only need to establish the estimate at $t = 0$. Let y be fixed in M^4. By rescaling, we can assume $R(y, 0) = 1$.

Let us first consider the case that $\sup\{R(x, 0)d_0^2(x, y) | x \in M^4\} > 1$. Define z to be the closest point to y (at time $t = 0$) satisfying $R(z, 0)d_0^2(z, y) = 1$. We want to bound $R(x, 0)/R(z, 0)$ from above for $x \in B_0(z, 2R(z, 0)^{-\frac{1}{2}})$.

Connect y to z by a shortest geodesic and choose a point $\widetilde{z}$ lying on the geodesic satisfying $d_0(\widetilde{z}, z) = \frac{1}{4}R(z, 0)^{-\frac{1}{2}}$. Denote by B the ball centered at $\widetilde{z}$ and with radius $\frac{1}{4}R(z, 0)^{-\frac{1}{2}}$ (with respect to the metric at $t = 0$). Clearly the ball B lies in $B_0(y, R(z, 0)^{-\frac{1}{2}})$ and lies outside $B_0(y, \frac{1}{2}R(z, 0)^{-\frac{1}{2}})$. Thus as $x \in B$, we have

$$R(x, 0)d_0^2(x, y) \leq 1 \text{ and } d_0(x, y) \geq \frac{1}{2}R(z, 0)^{-\frac{1}{2}},$$

which imply

$$R(x, 0) \leq \frac{1}{(\frac{1}{2}R(z, 0)^{-\frac{1}{2}})^2}, \text{ on } B.$$

Then by Li-Yau-Hamilton inequality [**18**] and the κ-noncollapsing, we have

$$\text{Vol}_0(B) \geq \kappa(\frac{1}{4}R(z, 0)^{-\frac{1}{2}})^4$$

and then

$$\text{Vol}_0(B_0(z, 8R(z, 0)^{-\frac{1}{2}}) \geq \frac{\kappa}{2^{20}}(8R(z, 0)^{-\frac{1}{2}})^4.$$

So by Corollary 11.6 of [**31**], there exists a positive constant A_1 depending only on κ such that

(3.1) $R(x, 0) \leq A_1 R(z, 0), \text{ for } x \in B_0(z, 2R(z, 0)^{-\frac{1}{2}}).$

We now consider the remaining case: $R(x,0)d_0^2(x,y) \leq 1$ for all $x \in M^4$. We choose a point $z \in M^4$ satisfying $R(z,0) \geq \frac{1}{2}\sup\{R(x,0)|x \in M^4\}$. Obviously we also have the estimate (3.1) in the remaining case.

After having the estimate (3.1), we next want to bound $R(z,0)$ for the chosen $z \in M^4$. By combining with Li-Yau-Hamilton inequality [18], we have

$$R(x,t) \leq A_1 R(z,0),$$

for all $x \in B_0(z, 2R(z,0)^{-\frac{1}{2}})$ and all $t \leq 0$. It then follows from Shi's local derivative estimate [36] that

$$\frac{\partial}{\partial t}R(z,t) \leq A_2 R(z,0)^2, \quad \text{for all } -R^{-1}(z,0) \leq t \leq 0,$$

where A_2 is some constant depending only on κ. This implies

$$R(z, -cR^{-1}(z,0)) \geq cR(z,0)$$

for some small positive constant c depending only on κ. On the other hand, by using the Harnack estimate [18] (as a consequence of Li-Yau-Hamilton inequality), we have

$$1 = R(y,0) \geq \widetilde{c}R(z, -cR^{-1}(z,0))$$

for some small positive constant $\widetilde{c}$ depending only on κ. Thus we obtain

(3.2) $$R(z,0) \leq A_3$$

for some positive constant A_3 depending only on κ.

The combination of (3.1) and (3.2) gives

$$R(x,0) \leq A_1 A_3, \text{ on } B_0\left(y, A_3^{-\frac{1}{2}}\right).$$

Thus by the κ-noncollapsing there exists a positive constant r_0 depending only on κ such that

$$\text{Vol}_0(B_0(y, r_0)) \geq \kappa r_0^4.$$

For any fixed $R_0 \geq r_0$, we have

$$\text{Vol}_0(B_0(y, R_0)) \geq \kappa \left(\frac{r_0}{R_0}\right)^4 \cdot R_0^4.$$

By applying Corollary 11.6 of [31] again, there exists a positive constant $\omega(R_0^2)$ depending only on R_0 and κ such that

$$R(x,0) \leq \omega(R_0^2), \text{ on } B_0\left(y, \frac{1}{4}R_0\right).$$

This gives the desired estimate.

(ii) It immediately follows from the above assertion (i), the Li-Yau-Hamilton inequality [18] and Shi's local derivative estimates [36]. q.e.d.

Remark. The argument in the last paragraph of the above proof for (i) implies the following assertion:

For any $\zeta > 0$, there is a positive function ω depending only on ζ such that if there holds

$$\frac{\mathrm{Vol}_{t_0}(B_{t_0}(y, R(y, t_0)^{-\frac{1}{2}}))}{R(y, t_0)^{-\frac{4}{2}}} \geq \zeta,$$

for some **fixed** point y and some $t_0 \in (-\infty, 0]$, then we have the following the elliptic type estimate

$$R(x, t_0) \leq R(y, t_0) \cdot \omega(R(y, t_0)d_{t_0}^2(x, y))$$

for all $x \in M$.

This estimate will play a key role in deriving the universal noncollapsing property in the next subsection.

Let $g_{ij}(t)$, $-\infty < t \leq 0$, be a nonflat solution to the Ricci flow on a four-manifold M^4. Fix a small $\varepsilon > 0$. We say that a point $x_0 \in M^4$ is the **center of an evolving ε-neck**, if the solution $g_{ij}(t)$ in the set $\{(x, t)| -\varepsilon^{-2}Q^{-1} < t \leq 0, d_0^2(x, x_0) < \varepsilon^{-2}Q^{-1}\}$(with $Q = R(x_0, 0)$) is, after scaling with factor Q, ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the corresponding subset of the evolving round cylinder $\mathbb{R} \times \mathbb{S}^3$, having scalar curvature one at $t = 0$.

The following result generalizes Corollary 11.8 of Perelman [**31**] to four-dimension and verifies Theorem E 3.3 of Hamilton [**21**]. The crucial information in the following Proposition is that the constant $C = C(\varepsilon) > 0$ depends **only** on ε.

Proposition 3.4. For any $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that if $g_{ij}(t)$ is a nonflat ancient κ-solution with restricted isotropic curvature pinching on a **noncompact** four-manifold M^4 for some $\kappa > 0$, and M_ε^4 denotes the set of points of M^4, which are not centers of evolving ε-necks, then either the whole M^4 is a metric quotient of the round cylinder $\mathbb{R} \times \mathbb{S}^3$ or M_ε^4 satisfies the following properties:

(i) M_ε^4 is compact, and

(ii) $\mathrm{diam}\,(M_\varepsilon^4) \leq CQ^{-\frac{1}{2}}$ and $C^{-1}Q \leq R(x, 0) \leq CQ$, whenever $x \in M_\varepsilon^4$, where $Q = R(x_0, 0)$ for some $x_0 \in \partial M_\varepsilon^4$ and $\mathrm{diam}\,(M_\varepsilon^4)$ is the diameter of the set M_ε^4 with respect to the metric $g_{ij}(0)$.

Proof. Note that the curvature operator of the ancient κ-solution is nonnegative. We first consider the easy case that the curvature operator has a nontrivial null vector somewhere at some time. By Lemma 3.2, we know that the ancient κ-solution is a metric quotient of the round cylinder $\mathbb{R} \times \mathbb{S}^3$.

We then assume the curvature operator of the ancient κ-solution is positive everywhere. Firstly we want to show that M_ε^4 is compact.

Argue by contradiction. Suppose there exists a sequence of points z_k, $k = 1, 2, \ldots$, going to infinity (with respect to the metric $g_{ij}(0)$) such that each z_k is not the center of any evolving ε-neck. For arbitrarily fixed point $z_0 \in M^4$, it follows from Proposition 3.3 (i) that

$$0 < R(z_0, 0) \leq R(z_k, 0) \cdot \omega(R(z_k, 0)d_0^2(z_k, z_0)),$$

which implies that

$$\lim_{k \to \infty} R(z_k, 0)d_0^2(z_k, z_0) = +\infty.$$

By Lemma 3.1 and Proposition 3.3 and Hamilton's compactness theorem, we conclude that z_k is the center of an evolving ε-neck as k sufficiently large. This is a contradiction, so we have proved that M_ε^4 is compact.

Note that M^4 is diffeomorphic to $\mathbb{R}^4$ since the curvature operator is positive. We may assume $\varepsilon > 0$ so small that Hamilton's replacement for Schoenflies conjecture and its proof (Theorem G1.1 and Lemma G1.3 of [**21**]) are available. Since every point outside the compact set M_ε^4 is the center of an evolving ε-neck, it follows that the approximate round three-sphere cross-section through the center divides M^4 into two connected components such that one of them is diffeomorphic to the four-ball $\mathbb{B}^4$. Let φ be a Busemann function on M^4(constructed from all geodesic rays emanating from a given point); it is a standard fact that φ is convex and proper. Since M_ε^4 is compact, M_ε^4 is contained in a compact set $K = \varphi^{-1}((-\infty, A])$ for some large A. We note that each point $x \in M^4 \setminus M_\varepsilon$ is the center of an ε-neck. It is clear that there is an ε-neck N lying entirely outside K. Consider a point x on one of its boundary components of the ε-neck N. Since $x \in M^4 \setminus M_\varepsilon^4$, there is an ε-neck adjacent to the initial ε-neck, producing a longer neck. We then take a point on the boundary of the second ε-neck and continue. This procedure can either terminate when we get into M_ε or go on infinitely to produce a semi-infinite (topological) cylinder. The same procedure can be repeated for the other boundary component of the initial ε-neck. This procedure will give a maximal extended neck $\tilde{N}$. If $\tilde{N}$ never touches M_ε^4, the manifold will be diffeomorphic to the standard infinite cylinder, which is a contradiction. If both of the two ends of $\tilde{N}$ touch M_ε^4, then there is a geodesic connecting two points of M_ε^4 and passing through N. This is impossible since the function φ is convex. So we conclude that one end of $\tilde{N}$ will touch M_ε^4 and the other end will tend to infinity to produce a semi-infinite (topological) cylinder. Then one can find an approximate round three-sphere cross-section which encloses the whole set M_ε^4 and touches some point $x_0 \in \partial M_\varepsilon^4$. We now want to show that $R(x_0, 0)^{\frac{1}{2}} \cdot \mathrm{diam}\,(M_\varepsilon^4)$ is bounded from above by some positive constant $C = C(\varepsilon)$ depending ***only*** on ε.

Suppose not; there exists a sequence of nonflat noncompact ancient κ_j-solutions with restricted isotropic curvature pinching and with positive curvature operator, for some sequence of positive constants κ_j, such that for the above chosen points $x_0 \in M_\varepsilon^4$ there would hold

$$(3.3) \qquad R(x_0, 0)^{\frac{1}{2}} \cdot \text{diam}\,(M_\varepsilon^4) \to +\infty.$$

Since the point x_0 lies in some 2ε-neck, clearly, there is a universal positive lower bound for $\text{Vol}_0(B_0(x_0, \frac{1}{\sqrt{R(x_0,0)}}))/(\frac{1}{\sqrt{R(x_0,0)}})^4$. By the remark after the proof of the previous Proposition 3.3, we see that there is a universal positive function $\omega : [0, +\infty) \to [0, +\infty)$ such that the elliptic type estimate

$$(3.4) \qquad R(x, 0) \leq R(x_0, 0) \cdot \omega(R(x_0, 0)d_0^2(x, x_0))$$

holds for all $x \in M$.

Let us scale the ancient solutions around the points x_0 with the factors $R(x_0, 0)$. By (3.4), Hamilton's compactness theorem (Theorem 16.1 of [20]) and the universal noncollapsing property at x_0, we can extract a convergent subsequence. From the choice of the points x_0 and (3.3), the limit contains a line. Actually we may draw a geodesic ray from some point $x_1 \in M_\varepsilon^4$ which is far from x_0 (in the normalized distance). This geodesic ray must cross some vertical three-sphere containing x_0. The limit of these rays gives us a line. Then by Toponogov splitting theorem the limit is isometric to $\mathbb{R} \times X^3$ for some smooth three-manifold X^3. As before, by using the restricted isotropic curvature pinching condition (2.4) and the contracted second Bianchi identity, we see that $X^3 = \mathbb{S}^3/\Gamma$ for some group Γ of isometrics without fixed points. Then we apply the same argument as in the proof of Lemma 3.1 to conclude that $\Gamma = \{1\}$. This says that the limit must be the evolving round cylinder $\mathbb{R} \times \mathbb{S}^3$. This contradicts with the fact that each chosen point x_0 is not the center of any evolving ε-neck. Therefore we have proved

$$\text{diam}\,(M_\varepsilon^4) \leq CQ^{-\frac{1}{2}}$$

for some positive constant $C = C(\varepsilon)$ depending only on ε, where $Q = R(x_0, 0)$.

Finally, by combining this diameter estimate and the remark after Proposition 3.3, we directly deduce

$$\widetilde{C}^{-1}Q \leq R(x, 0) \leq \widetilde{C}Q, \text{ whenever } x \in M_\varepsilon^4,$$

for some positive constant $\widetilde{C}$ depending only on ε. \hfill q.e.d.

Consequently, by applying the standard volume comparison to Proposition 3.4, we conclude that all complete ***noncompact*** four-dimensional

ancient κ-solutions with restricted isotropic curvature pinching and positive curvature operator are κ_0-noncollapsing on all scales for some universal constant $\kappa_0 > 0$. In the next subsection, we will prove this universal noncollapsing property for both compact and noncompact cases.

3.3. Universal noncollapsing of ancient κ-solutions. First we note that the universal noncollapsing is not true for all metric quotients of round $\mathbb{R} \times \mathbb{S}^3$. The main result of this section is to establish the universal noncollapsing property for all ancient κ-solutions with restricted isotropic curvature pinching which are not metric quotients of round $\mathbb{R} \times \mathbb{S}^3$. The analogous result for three-dimensional ancient κ-solutions was claimed by Perelman in Remark 11.9 of [**31**] and Section 1.5 of [**32**].

Theorem 3.5. *There exists a positive constant κ_0 with the following property. Suppose we have a four-dimensional* (*compact or noncompact*) *ancient κ-solution with restricted isotropic curvature pinching for some $\kappa > 0$. Then either the solution is κ_0-noncollapsed for all scales, or it is a metric quotient of the round cylinder $\mathbb{R} \times \mathbb{S}^3$.*

Proof. Let $g_{ij}(x,t)$, $x \in M^4$ and $t \in (-\infty, 0]$, be an ancient κ-solution with restricted isotropic curvature pinching for some $\kappa > 0$. We had known that the curvature operator of the solution $g_{ij}(x,t)$ is nonnegative everywhere and every time. If the curvature operator of the solution $g_{ij}(x,t)$ has a nontrivial null eigenvector somewhere at some time, then we know from Lemma 3.2 that the solution is a metric quotient of the round neck $\mathbb{R} \times \mathbb{S}^3$.

We now assume the solution $g_{ij}(x,t)$ has positive curvature operator everywhere and every time. We want to apply the backward limit argument of Perelman to take a sequence of points q_k and a sequence of times $t_k \to -\infty$ such that the scalings of $g_{ij}(\cdot, t)$ around q_k with factors $|t_k|^{-1}$ (and shifting the times t_k to zero) converge in C^∞_{loc} topology to a non-flat gradient shrinking soliton.

Clearly, we may assume the nonflat ancient κ-solution is not a gradient shrinking Ricci soliton. For arbitrary point $(p, t_0) \in M^4 \times (-\infty, 0]$, we define as in [**31**] that

$$\tau = t_0 - t, \text{ for } t < t_0,$$

$$l(q, \tau) = \frac{1}{2\sqrt{\tau}} \inf \left\{ \int_0^\tau \sqrt{s}(R(\gamma(s), t_0 - s) + |\dot{\gamma}(s)|^2_{g_{ij}(t_0-s)})ds| \right.$$

$$\left. \gamma : [0, \tau] \to M^4 \text{ with } \gamma(0) = p, \gamma(\tau) = q \right\},$$

and $\widetilde{V}(\tau) = \int_{M^4} (4\pi\tau)^{-2} \exp(-l(q, \tau))dV_{t_0-\tau}(q),$

where $|\cdot|_{g_{ij}(t_0-s)}$ is the norm with respect to the metric $g_{ij}(t_0 - s)$ and $dV_{t_0-\tau}$ is the volume element with respect to the metric $g_{ij}(t_0 - \tau)$.

According to [31], l is called the **reduced distance** and $\widetilde{V}(\tau)$ is called the **reduced volume**. Since the manifold M^4 may be noncompact, one would ask whether the reduced volume is finite. Since the scalar curvature is nonnegative and the curvature is bounded, it is not hard to see that the reduced distance is quadratically grown and that the reduced volume is always finite. (Actually, by using Perelman's Jacobian comparison theorem [31] one can show that the reduced volume is always finite for any complete solution of the Ricci flow (see [3] for the details)). In [31], Perelman proved that the reduced volume $\widetilde{V}(\tau)$ is nonincreasing in τ, and the monotonicity is strict unless the solution is a gradient shrinking Ricci soliton.

From [31] (Section 7 of [31]), the function $\overline{L}(q, \tau) = 4\tau l(q, \tau)$ satisfies

$$\frac{\partial}{\partial \tau}\overline{L} + \triangle \overline{L} \leq 8.$$

It is clear that $\overline{L}(\cdot, \tau)$ achieves its minimum on M^4 for each $\tau > 0$ since the scalar curvature is nonnegative. Then the minimum of $\overline{L}(\cdot, \tau) - 8\tau$ is nonincreasing, so in particular, the minimum of $l(\cdot, \overline{\tau})$ does not exceed 2 for each $\tau > 0$. Thus for each $\tau > 0$ we can find $q = q(\tau)$ such that $l(q(\tau), \tau) \leq 2$. We can apply Perelman's Proposition 11.2 from [31] to conclude that the scalings of $g_{ij}(\cdot, t_0 - \tau)$ around $q(\tau)$ with factors τ^{-1} converge in C_{loc}^{∞} topology along a subsequence $\tau \to +\infty$ to a non-flat gradient shrinking soliton. Because the proof of this proposition in [31] is just a sketch, we would like to give its detail in the following for completeness.

We first claim that for any $A \geq 1$, one can find $B = B(A) < +\infty$ such that for every $\overline{\tau} > 1$ there holds

(3.5) $$l(q, \tau) < B \text{ and } \tau R(q, t_0 - \tau) \leq B$$

whenever $\frac{1}{2}\overline{\tau} \leq \tau \leq A\overline{\tau}$ and $d^2_{t_0 - \frac{\overline{\tau}}{2}}(q, q(\frac{\overline{\tau}}{2})) \leq A\overline{\tau}$.

Indeed, by Section 7 of [31], the reduced distance l satisfies the following

(3.6)
(3.7)
(3.8)
$$\begin{cases} \frac{\partial}{\partial \tau}l = -\frac{l}{\tau} + R + \frac{K}{2\tau^{3/2}}, \\ |\nabla l|^2 = -R + \frac{l}{\tau} - \frac{K}{\tau^{3/2}}, \\ \triangle l \leq -R + \frac{2}{\tau} - \frac{K}{2\tau^{3/2}}, \end{cases}$$

in the sense of distributions, and the equality holds everywhere if and only if we are on a gradient shrinking soliton, where $K = \int_0^\tau s^{3/2} Q(X) ds$ and $Q(X)$ is the trace Li-Yau-Hamilton quadratic given by

$$Q(X) = -\frac{\partial}{\partial \tau}R - \frac{R}{\tau} - 2\langle \nabla R, X \rangle + 2Ric(X, X),$$

and X is the tangential (velocity) vector of a $\mathcal{L}$-shortest curve $\gamma : [0, \tau] \to M^4$ connecting p to q.

By applying the trace Li-Yau-Hamilton inequality [**18**] to the ancient κ-solution, we have

$$Q(X) \geq -\frac{R}{\tau}$$

and then

$$K \geq -\int_0^\tau \sqrt{s}Rds \geq -2\sqrt{\tau}l.$$

Thus by (3.7) we get

$$(3.9) \qquad\qquad |\nabla l|^2 + R \leq \frac{3l}{\tau}.$$

At $\tau = \frac{\bar{\tau}}{2}$, we have

$$(3.10) \qquad \sqrt{l\left(q, \frac{\bar{\tau}}{2}\right)} \leq \sqrt{2} + \sup\{|\nabla \sqrt{l}|\} \cdot d_{t_0 - \frac{\bar{\tau}}{2}}\left(q, q\left(\frac{\bar{\tau}}{2}\right)\right)$$

$$\leq \sqrt{2} + \sqrt{\frac{3A}{2}},$$

and

$$(3.11) \qquad R\left(q, t_0 - \frac{\bar{\tau}}{2}\right) \leq \frac{6}{\bar{\tau}}\left(\sqrt{2} + \sqrt{\frac{3A}{2}}\right)^2,$$

for $q \in B_{t_0 - \frac{\bar{\tau}}{2}}(q(\frac{\bar{\tau}}{2}), \sqrt{A\bar{\tau}})$. Since the scalar curvature of an ancient solution with nonnegative curvature operator is pointwisely nondecreasing in time (by the trace Li-Yau-Hamilton inequality [**18**]), we further have

$$(3.12) \qquad \tau R(q, t_0 - \tau) \leq 6A\left(\sqrt{2} + \sqrt{\frac{3A}{2}}\right)^2$$

whenever $\frac{1}{2}\bar{\tau} \leq \tau \leq A\bar{\tau}$ and $d^2_{t_0 - \frac{\bar{\tau}}{2}}(q, q(\frac{\bar{\tau}}{2})) \leq A\bar{\tau}$.

By (3.6), (3.7) and (3.12), we have

$$\frac{\partial}{\partial \tau}l \leq -\frac{l}{2\tau} + \frac{3A}{\tau}\left(\sqrt{2} + \sqrt{\frac{3A}{2}}\right)^2,$$

and by integrating this inequality and using the estimate (3.10), we obtain

$$(3.13) \qquad l(q, \tau) \leq 7A\left(\sqrt{2} + \sqrt{\frac{3A}{2}}\right)^2$$

whenever $\frac{1}{2}\bar{\tau} \leq \tau \leq A\bar{\tau}$ and $d^2_{t_0 - \frac{\bar{\tau}}{2}}(q, q(\frac{\bar{\tau}}{2})) \leq A\bar{\tau}$. So we have proved the assertion (3.5).

The scaling of the ancient κ-solution around $q(\frac{\overline{\tau}}{2})$ with factor $(\frac{\overline{\tau}}{2})^{-1}$ is

$$\widetilde{g}_{ij}(s) = \frac{2}{\overline{\tau}}g_{ij}\left(\cdot, t_0 - s\frac{\overline{\tau}}{2}\right)$$

for $s \in [0, +\infty)$. The assertion (3.5) implies that for all $s \in [1, 2A]$ and all q with $dist^2_{\widetilde{g}_{ij}(1)}(q, q(\frac{\overline{\tau}}{2})) \leq A$, we have $\widetilde{R}(q,s) \leq B$ where $\widetilde{R}$ is the scalar curvature of the rescaled metric $\widetilde{g}_{ij}$. Then we can use Hamilton's compactness theorem ([**19**] or more precisely Theorem 16.1 of [**20**]) and the κ-noncollapsing assumption to obtain a sequence $\overline{\tau}_k \to +\infty$ such that the marked evolving manifolds $(M^4, \widetilde{g}_{ij}^{(k)}(s), q(\frac{\overline{\tau}_k}{2}))$, with $\widetilde{g}_{ij}^{(k)}(s) = \frac{2}{\overline{\tau}_k}g_{ij}(\cdot, t_0 - s\frac{\overline{\tau}_k}{2})$ and $s \in [1, +\infty)$, converge in C^∞_{loc} topology to an evolving manifold $(\overline{M}^4, \overline{g}_{ij}(s), \overline{q})$ with $s \in [1, +\infty)$, where $\overline{g}_{ij}(s)$ satisfies $\frac{\partial}{\partial s}\overline{g}_{ij} = 2\overline{R}_{ij}$ on $\overline{M} \times [1, +\infty)$.

Denote by $\widetilde{l}_k$ the corresponding reduced distance of $\widetilde{g}_{ij}^{(k)}(s)$. It is easy to see that $\widetilde{l}_k(q, s) = l(q, \frac{\overline{\tau}_k}{2}s)$ for $s \in [1, +\infty)$. After rescaling we still have

$$|\nabla\widetilde{l}_k|^2_{\widetilde{g}_{ij}^{(k)}} + \widetilde{R}^{(k)} \leq 6\widetilde{l}_k$$

and by (3.5), $\widetilde{l}_k$ are uniformly bounded at finite distances. Thus the above gradient estimate implies that the functions $\widetilde{l}_k$ tend (up to a subsequence) to a function $\overline{l}$ which is a locally Lipschitz function on $\overline{M}$.

From (3.6)-(3.8), we have

$$\frac{\partial}{\partial s}(\widetilde{l}_k) - \triangle\widetilde{l}_k + |\nabla\widetilde{l}_k|^2 - \widetilde{R}^{(k)} + \frac{2}{s} \geq 0,$$

$$2\triangle\widetilde{l}_k - |\nabla\widetilde{l}_k|^2 + \widetilde{R}^{(k)} + \frac{\widetilde{l}_k - 4}{s} \leq 0,$$

which can be rewritten as

(3.14) $$\left(\frac{\partial}{\partial s} - \triangle + \widetilde{R}^{(k)}\right)((4\pi s)^{-2}\exp(-\widetilde{l}_k)) \leq 0,$$

(3.15) $$-(4\triangle - \widetilde{R}^{(k)})e^{-\frac{\widetilde{l}_k}{2}} + \frac{\widetilde{l}_k - 4}{s}e^{-\frac{\widetilde{l}_k}{2}} \leq 0,$$

in the sense of distribution. Clearly, these two inequalities imply that the limit $\overline{l}$ satisfies

(3.16) $$\left(\frac{\partial}{\partial s} - \triangle + \overline{R}\right)((4\pi s)^{-2}\exp(-\overline{l})) \leq 0,$$

(3.17) $$-(4\triangle - \overline{R})e^{-\frac{\overline{l}}{2}} + \frac{\overline{l} - 4}{s}e^{-\frac{\overline{l}}{2}} \leq 0,$$

in the sense of distribution.

Denote by $\widetilde{V}^{(k)}(s)$ the reduced volume of the rescaled metric $\widetilde{g}_{ij}^{(k)}(s)$. Since $\widetilde{l}_k(q,s) = l(q, \frac{\overline{\tau}_k}{2}s)$, we see that $\widetilde{V}^{(k)}(s) = \widetilde{V}(\frac{\overline{\tau}_k}{2}s)$. The monotonicity of the reduced volume $\widetilde{V}(\tau)$ (see [**31**]) then implies that

$$\lim_{k \to +\infty} \widetilde{V}^{(k)}(s) = \overline{V}, \text{ for } s \in [1,2]$$

for some positive constant $\overline{V}$. But we are not sure whether the limiting $\overline{V}$ is exactly Perelman's reduced volume of the limiting manifold $(\overline{M}^4, \overline{g}_{ij}(s))$, because the points $q(\frac{\overline{\tau}_k}{2})$ may diverge to infinity. Nevertheless, we can insure that $\overline{V}$ is not less than Perelman's reduced volume of the limit. Note that

$$\widetilde{V}^{(k)}(2) - \widetilde{V}^{(k)}(1)$$
$$= \int_1^2 \frac{d}{ds}(\widetilde{V}^{(k)}(s))ds$$
$$= \int_1^2 ds \int_{M^4} \left(\frac{\partial}{\partial s} - \triangle + \widetilde{R}^{(k)}\right)((4\pi s)^{-2}\exp(-\widetilde{l}_k))dV_{\widetilde{g}_{ij}^{(k)}(s)}.$$

Thus we deduce that in the sense of distributions,

$$(3.18) \qquad \left(\frac{\partial}{\partial s} - \triangle + \overline{R}\right)((4\pi s)^{-2}\exp(-\overline{l})) = 0,$$

$$(3.19) \qquad -(4\triangle - \overline{R})e^{-\frac{\overline{l}}{2}} + \frac{\overline{l}-4}{s}e^{-\frac{\overline{l}}{2}} = 0,$$

and then the standard parabolic equation theory implies that $\overline{l}$ is actually smooth. Here we used (3.6)-(3.8) to show that the equality in (3.16) implies the equality in (3.17).

Set

$$\upsilon = [s(2\triangle\overline{l} - |\nabla\overline{l}|^2 + \overline{R}) + \overline{l} - 4] \cdot (4\pi s)^{-2}e^{-\overline{l}}.$$

A direct computation gives

$$(3.20) \quad \left(\frac{\partial}{\partial s} - \triangle + \overline{R}\right)\upsilon = -2s|\overline{R}_{ij} + \nabla_i\nabla_j\overline{l} - \frac{1}{2s}\overline{g}_{ij}|^2 \cdot (4\pi s)^{-2}e^{-\overline{l}}.$$

Since the equation (3.18) implies $\upsilon \equiv 0$, the limit metric $\overline{g}_{ij}$ satisfies

$$(3.21) \qquad \overline{R}_{ij} + \nabla_i\nabla_j\overline{l} - \frac{1}{2s}\overline{g}_{ij} = 0.$$

Thus the limit is a gradient shrinking Ricci soliton.

To show the limiting gradient shrinking Ricci soliton to be nonflat, we first show that constant $\overline{V}$ is strictly less than 1. Indeed, by considering the reduced volume $\widetilde{V}(\tau)$ of the ancient κ-solution, we get from

Perelman's Jacobian comparison theorem [**31**] that

$$
\begin{aligned}
\widetilde{V}(\tau) &= \int_{M^4} (4\pi\tau)^{-2} e^{-l} dV_{t_0 - \tau} \\
&\leq \int_{T_p M^4} (4\pi)^{-2} e^{-|X|^2} dX \\
&= 1.
\end{aligned}
$$

Recall that we assumed the nonflat ancient κ-solution is not a gradient shrinking Ricci soliton. Thus by the monotonicity of the reduced volume [**31**], we have $\widetilde{V}(\tau) < 1$ for $\tau > 0$. This implies that $\overline{V} < 1$.

We now argue by contradiction. Suppose the limit $\overline{g}_{ij}(s)$ is flat. Then by (3.21) we have

$$
\nabla_i \nabla_j \overline{l} = \frac{1}{2s} \overline{g}_{ij} \quad \text{and} \quad \triangle \overline{l} = \frac{2}{s}.
$$

And then by (3.19), we get

$$
|\nabla \overline{l}|^2 = \frac{\overline{l}}{s}.
$$

Since the function $\overline{l}$ is strictly convex, it follows that $\sqrt{4s\overline{l}}$ is a distance function (from some point) on the complete flat manifold $\overline{M}$. From the smoothness of the function $\overline{l}$, we conclude that the flat manifold $\overline{M}$ must be $\mathbb{R}^4$. In this case we would have its reduced distance to be $\overline{l}$ and its reduced volume to be 1. Since $\overline{V}$ is not less than the reduced volume of the limit, this is a contradiction. Therefore the limiting gradient shrinking soliton $\overline{g}_{ij}$ is nonflat.

Now we consider the nonflat gradient shrinking Ricci soliton $(\overline{M}^4, \overline{g}_{ij})$. Of course it is still κ-noncollapsed for all scales and satisfies the restricted isotropic curvature pinching condition (2.4). We first show that $(\overline{M}^4, \overline{g}_{ij}(s))$ has bounded curvature at each $s > 0$. Clearly it suffices to consider $s = 1$. By Lemma 3.2, we may assume the soliton $(\overline{M}^4, \overline{g}_{ij}(1))$ has positive curvature operator everywhere. Let us argue by contradiction. Suppose not; then we claim that for each positive integer k, there exists a point x_k such that

$$
\begin{cases}
\overline{R}(x_k, 1) \geq k, \\
\overline{R}(x, 1) \leq 4\overline{R}(x_k, 1), \quad \text{for } x \in B_{\overline{g}(\cdot, 1)}\left(x_k, \frac{k}{\sqrt{\overline{R}(x_k, 1)}} \right).
\end{cases}
$$

Indeed, x_k can be constructed as a limit of a finite sequence $\{y_i\}$, defined as follows. Let y_0 be any fixed point with $\overline{R}(y_0, 1) \geq k$. Inductively, if y_i cannot be taken as x_k, then there is a y_{i+1} such that

$$
\begin{cases}
\overline{R}(y_{i+1}, 1) > 4\overline{R}(y_i, 1), \\
d_{\overline{g}(\cdot, 1)}(y_i, y_{i+1}) \leq \frac{k}{\sqrt{\overline{R}(y_i, 1)}}.
\end{cases}
$$

Thus we have

$$\bar{R}(y_i, 1) > 4^i \bar{R}(y_0, 1) \geq 4^i k,$$

$$d_{\bar{g}(\cdot, 1)}(y_i, y_0) \leq k \sum_{j=1}^{i} \frac{1}{\sqrt{4^{j-1}k}} < 2\sqrt{k}.$$

Since the soliton is smooth, the sequence $\{y_i\}$ must be finite. The last element fits.

Note that the limiting soliton still satisfies the Li-Yau-Hamilton inequality. Then we have

$$\bar{R}(x, s) \leq \bar{R}(x, 1) \leq 4\bar{R}(x_k, 1)$$

for $x \in B_{\bar{g}(\cdot, 1)}(x_k, \frac{k}{\sqrt{\bar{R}(x_k, 1)}})$ and $1 \leq s \leq 1 + \frac{1}{\bar{R}(x_k, 1)}$. By the κ-noncollapsing and the Hamilton's compactness theorem [**19**], a sequence of $(\bar{M}^4, \bar{R}(x_k, 1)\bar{g}(\cdot, 1 + \frac{(\cdot)}{\bar{R}(x_k, 1)}), x_k)$ will converge to a complete smooth solution $(\bar{\bar{M}}^4, \bar{\bar{g}})$ at least on the interval $[0, 1)$. Since $d_{\bar{g}(\cdot, 1)}(x_k, x_0) \to \infty$ and $\bar{R}(x_k, 1) \to \infty$, it follows from Lemma 3.1 that $\bar{\bar{M}}^4 = \mathbb{R} \times \mathbb{S}^3$. This contradicts Proposition 2.2. So we have proved that $(\overline{M}^4, \overline{g}_{ij}(s))$ has bounded curvature at each $s > 0$.

We next show that the soliton $(\overline{M}^4, \overline{g}_{ij})$ is κ'_0-noncollapsed on all scales for some universal positive constant κ'_0. If the soliton $(\overline{M}^4, \overline{g}_{ij})$ has positive curvature operator, we know from Hamilton's result [**16**] and Proposition 3.4 that either the soliton $(\overline{M}^4, \overline{g}_{ij})$ is the round $\mathbb{S}^4$ or $\mathbb{RP}^4$ when it is compact, or it is κ'_0-noncollapsed for all scales for some universal positive constant κ'_0 when it is noncompact. (Furthermore, when the soliton $(\overline{M}^4, \overline{g}_{ij})$ is the round $\mathbb{S}^4$ or $\mathbb{RP}^4$, it follows from Hamilton's pinching estimates in [**16**] that the original ancient κ-solution $(M^4, g_{ij}(t))$ is also the round $\mathbb{S}^4$ or $\mathbb{RP}^4$). While if the soliton $(\overline{M}^4, \overline{g}_{ij})$ has a nontrivial null eigenvector somewhere at some time, we know from Lemma 3.2 that the soliton $(\overline{M}^4, \overline{g}_{ij})$ is $\mathbb{R} \times \mathbb{S}^3/\Gamma$, a metric quotient of the round neck $\mathbb{R} \times \mathbb{S}^3$. For each $\sigma \in \Gamma$, $(s, x) \in \mathbb{R} \times \mathbb{S}^3$, write $\sigma(s, x) = (\sigma_1(s, x), \sigma_2(s, x)) \in \mathbb{R} \times \mathbb{S}^3$. Since σ sends lines to lines, and σ sends cross spheres to cross spheres, we have $\sigma_1(s, x) = \sigma_1(s, y), \forall x, y \in \mathbb{S}^3$. This says that σ_1 reduces to a function of s alone on $\mathbb{R}$. Moreover, for any $(s, x), (s', x') \in \mathbb{R} \times \mathbb{S}^3$, since σ preserves the distances between cross spheres $\{s\} \times \mathbb{S}^3$ and $\{s'\} \times \mathbb{S}^3$, we have $|\sigma_1(s, x) - \sigma_1(s', x')| = |s - s'|$. So the projection Γ_1 of Γ to the factor $\mathbb{R}$ is an isometric subgroup of $\mathbb{R}$. We know that if $(\bar{M}^4, \bar{g}_{ij}) = \mathbb{R} \times \mathbb{S}^3/\Gamma$ was compact, it, as an ancient solution, could not be κ-noncollapsed on all scales as $t \to -\infty$. Thus $(\bar{M}^4, \bar{g}_{ij}) = \mathbb{R} \times \mathbb{S}^3/\Gamma$ is noncompact. It follows that $\Gamma_1 = \{1\}$ or $\mathbb{Z}_2$. We conclude that, in both cases, there is a Γ-invariant cross sphere $\mathbb{S}^3$ in $\mathbb{R} \times \mathbb{S}^3$. Denote it by $\{0\} \times \mathbb{S}^3$. Γ acts on $\{0\} \times \mathbb{S}^3$ without fixed

points. Recall that we have assumed that the ancient solution (M^4, g_{ij}) has positive curvature operator. Then we apply Hamilton's argument in Theorem C4.1 of [**21**] when M^4 is compact and apply the modified argument in the proof of Lemma 3.1 when M^4 is noncompact to conclude that $(\{0\} \times \mathbb{S}^3)/\Gamma$ is incompressible in M^4 (i.e., $\pi_1((\{0\} \times \mathbb{S}^3)/\Gamma)$ injects into $\pi_1(M^4)$). By Synge theorem and the Soul theorem [**7**], the fundamental group $\pi_1(M^4)$ is either $\{1\}$ or $\mathbb{Z}_2$. This implies that Γ is either $\{1\}$ or $\mathbb{Z}_2$. Thus the limiting soliton $(\overline{M}^4, \overline{g}_{ij})$ is also κ_0'-noncollapsed on all scales for some universal positive constant κ_0'.

We next use the κ_0'-noncollapsing of the limiting soliton to derive a κ_0-noncollapsing for the original ancient κ-solution. By rescaling, we may assume that $R(x,t) \leq 1$ for all (x,t) satisfying $d_{t_0}(x,p) \leq 2$ and $t_0 - 1 \leq t \leq t_0$. We only need to bound the volume $\mathrm{Vol}_{t_0}(B_{t_0}(p,1))$ from below by a universal positive constant.

Denote by $\epsilon = \mathrm{Vol}_{t_0}(B_{t_0}(p,1))^{\frac{1}{4}}$. For any $v \in T_p M^4$, it is known from [**31**] that one can find a $\mathcal{L}$-geodesic $\gamma(\tau)$, starting at p, with $\lim_{\tau \to 0^+} \sqrt{\tau} \dot{\gamma}(\tau) = v$, which satisfies the following $\mathcal{L}$-geodesic equation

$$(3.22) \qquad \frac{d}{d\tau}(\sqrt{\tau} \dot{\gamma}) - \frac{1}{2}\sqrt{\tau} \nabla R + 2Ric(\sqrt{\tau}\dot{\gamma}, \cdot) = 0.$$

Note from Shi's local derivative estimate (see [**36**]) that $|\nabla R|$ is also uniformly bounded. By integrating the $\mathcal{L}$-geodesic equation we see that as $\tau \leq \epsilon$ with the property that $\gamma(\sigma) \in B_{t_0}(p,1)$ for $\sigma \in (0,\tau]$, there holds

$$(3.23) \qquad |\sqrt{\tau}\dot{\gamma}(\tau) - v| \leq C\epsilon(|v| + 1)$$

for some universal positive constant C. Here we implicitly used the fact that the metrics $g_{ij}(t)$ are equivalent to each other on $B_{t_0}(p,1) \times [t_0 - 1, t_0]$, which is an easy consequence of the boundedness of the curvature there. Without loss of generality, we may assume $C\epsilon \leq \frac{1}{4}$ and $\epsilon \leq \frac{1}{100}$. Then for $v \in T_p M^4$ with $|v| \leq \frac{1}{4}\epsilon^{-\frac{1}{2}}$ and for $\tau \leq \epsilon$ with the property that $\gamma(\sigma) \in B_{t_0}(p,1)$ for $\sigma \in (0,\tau]$, we have

$$\begin{aligned} d_{t_0}(p, \gamma(\tau)) &\leq \int_0^\tau |\dot{\gamma}(\sigma)| d\sigma \\ &< \frac{1}{2}\epsilon^{-\frac{1}{2}} \int_0^\tau \frac{d\sigma}{\sqrt{\sigma}} \\ &= 1. \end{aligned}$$

This shows

$$(3.24) \qquad \mathcal{L}\exp\left\{|v| \leq \frac{1}{4}\epsilon^{-\frac{1}{2}}\right\}(\epsilon) \subset B_{t_0}(p,1)$$

where $\mathcal{L}\exp(\cdot)(\epsilon)$ denotes the exponential map of the $\mathcal{L}$ distance with parameter ϵ (see [**31**] or [**3**] for details). We decompose the reduced

volume $\widetilde{V}(\epsilon)$ as

(3.25)

$$\widetilde{V}(\epsilon) = \int_{M^4} (4\pi\epsilon)^{-2} \exp(-l)dV_{t_0-\epsilon}$$

$$\leq \int_{\mathcal{L}\exp\{|v|\leq\frac{1}{4}\epsilon^{-\frac{1}{2}}\}(\epsilon)} + \int_{M^4\backslash\mathcal{L}\exp\{|v|\leq\frac{1}{4}\epsilon^{-\frac{1}{2}}\}(\epsilon)} (4\pi\epsilon)^{-2} \exp(-l)dV_{t_0-\epsilon}.$$

The first term on RHS of (3.25) can be estimated by

(3.26)

$$\int_{\mathcal{L}\exp\{|v|\leq\frac{1}{4}\epsilon^{-\frac{1}{2}}\}(\epsilon)} (4\pi\epsilon)^{-2} \exp(-l)dV_{t_0-\epsilon}$$

$$\leq e^{4\epsilon} \int_{B_{t_0}(p,1)} (4\pi\epsilon)^{-2} \exp(-l)dV_{t_0}$$

$$\leq e^{4\epsilon}(4\pi)^{-2}\epsilon^{-2}\mathrm{Vol}_{t_0}(B_{t_0}(p,1))$$

$$= e^{4\epsilon}(4\pi)^{-2}\epsilon^2,$$

where we used (3.24) and the equivalence of the evolving metric over $B_{t_0}(p,1)$. Meanwhile the second term on the RHS of (3.25) can be estimated as follows

(3.27)

$$\int_{M^4\backslash\mathcal{L}\exp\{|v|\leq\frac{1}{4}\epsilon^{-\frac{1}{2}}\}(\epsilon)} (4\pi\epsilon)^{-2} \exp(-l)dV_{t_0-\epsilon}$$

$$\leq \int_{\{|v|>\frac{1}{4}\epsilon^{-\frac{1}{2}}\}} (4\pi\tau)^{-2} \exp(-l)J(\tau)|_{\tau=0}dv$$

by Perelman's Jacobian comparison theorem [**31**], where $J(\tau)$ is the Jacobian of the $\mathcal{L}$-exponential map.

For any $v \in T_pM$, we consider a $\mathcal{L}$-geodesic $\gamma(\tau)$ starting at p with $\lim_{\tau\to 0^+} \sqrt{\tau}\dot\gamma(\tau) = v$. To evaluate the Jacobian of the $\mathcal{L}$ exponential map at $\tau = 0$ we choose linear independent vectors $v_1, \ldots, v_4$ in T_pM and let

$$V_i(\tau) = (\mathcal{L}\exp_v(\tau))_*(v_i) = \frac{d}{ds}|_{s=0}\mathcal{L}\exp_{(v+sv_i)}(\tau), \ i = 1, \ldots, 4.$$

The $\mathcal{L}$-Jacobian $J(\tau)$ is given by

$$J(\tau) = |V_1(\tau) \wedge \cdots \wedge V_4(\tau)|_{g_{ij}(\tau)}/|v_1 \wedge \cdots \wedge v_4|.$$

By the $\mathcal{L}$-geodesic equation (3.22) and the deriving of (3.23), we see that as $\tau > 0$ small enough,

$$\left|\sqrt{\tau}\frac{d}{d\tau}\mathcal{L}\exp_{(v+sv_i)}(\tau) - (v + sv_i)\right| \leq o(1)$$

for $s \in (-\epsilon, \epsilon)$ and $i = 1, \ldots, 4$, where $o(1)$ tends to zero as $\tau \to 0^+$ uniformly in s. This implies that

$$\lim_{\tau\to 0^+} \sqrt{\tau}\dot{V}_i(\tau) = v_i, \ i = 1, \ldots, 4,$$

so we get

$$(3.28) \qquad \lim_{\tau \to 0^+} \tau^{-2} J(\tau) = 1.$$

To evaluate $l(\cdot, \tau)$ at $\tau = 0$, we use (3.23) again to get

$$l(\cdot, \tau) = \frac{1}{2\sqrt{\tau}} \int_0^\tau \sqrt{s}(R + |\dot{\gamma}(s)|^2)ds$$

$$\to |v|^2, \qquad \text{as } \tau \to 0^+,$$

thus

$$(3.29) \qquad l(\cdot, 0) = |v|^2.$$

Hence by combining (3.27)-(3.29) we have

$$(3.30) \qquad \int_{M^4 \setminus \mathcal{L} \exp\{|v| \le \frac{1}{4}\epsilon^{-\frac{1}{2}}\}(\epsilon)} (4\pi\epsilon)^{-2} \exp(-l) dV_{t_0 - \epsilon}$$

$$\le (4\pi)^{-2} \int_{\{|v| > \frac{1}{4}\epsilon^{-\frac{1}{2}}\}} \exp(-|v|^2) dv$$

$$< \epsilon^2.$$

By summing up (3.25), (3.26) and (3.30), we obtain

$$(3.31) \qquad \widetilde{V}(\epsilon) < 2\epsilon^2.$$

On the other hand we recall that there are sequences $\tau_k \to +\infty$ and $q(\tau_k) \in M^4$ with $l(q(\tau_k), \tau_k) \le 2$ so that the rescalings of the ancient κ-solution around $q(\tau_k)$ with factor τ_k^{-1} converge to a gradient shrinking Ricci soliton which is κ_0'-noncollapsing on all scales for some universal positive constant κ_0'. For sufficiently large k, we construct a path $\gamma : [0, 2\tau_k] \to M^4$, connecting p to any given point $q \in M^4$, as follows: the first half path $\gamma|_{[0,\tau_k]}$ connects p to $q(\tau_k)$ such that

$$l(q(\tau_k), \tau_k) = \frac{1}{2\sqrt{\tau_k}} \int_0^{\tau_k} \sqrt{\tau}(R + |\dot{\gamma}(\tau)|^2)d\tau \le 3,$$

and the second half path $\gamma|_{[\tau_k, 2\tau_k]}$ is a shortest geodesic connecting $q(\tau_k)$ to q with respect to the metric $g_{ij}(t_0 - \tau_k)$. Note that the rescaled metric $\tau_k^{-1} g_{ij}(t_0 - \tau)$ over the domain $B_{t_0 - \tau_k}(q(\tau_k), \sqrt{\tau_k}) \times [t_0 - 2\tau_k, t_0 - \tau_k]$ is sufficiently close to the gradient shrinking Ricci soliton. Then by the estimates (3.5) and the κ_0'-noncollapsing of the shrinking soliton, we get

$$\widetilde{V}(2\tau_k) = \int_M (4\pi(2\tau_k))^{-2} \exp(-l(q, 2\tau_k)) dV_{t_0 - 2\tau_k}(q)$$

$$\ge \int_{B_{t_0 - \tau_k}(q(\tau_k), \sqrt{\tau_k})} (4\pi(2\tau_k))^{-2} \exp(-l(q, 2\tau_k)) dV_{t_0 - 2\tau_k}(q)$$

$$\ge \beta$$

for some universal positive constant β. By applying the monotonicity of the reduced volume [**31**] and (3.31), we deduce that

$$\beta \leq \widetilde{V}(2\tau_k) \leq \widetilde{V}(\epsilon) < 2\epsilon^2.$$

This proves

$$\mathrm{Vol}_{t_0}(B_{t_0}(p,1)) \geq \kappa_0 > 0$$

for some universal positive constant κ_0. Therefore we have proved the theorem.   q.e.d.

Once the universal noncollapsing of ancient κ-solution with restricted isotropic curvature pinching is established, we can also strengthen the elliptic type estimates in Proposition 3.3 to the following form.

Proposition 3.6. *There exist a positive constant η and a positive function $\omega : [0, +\infty) \to (0, +\infty)$ with the following properties. Suppose we have a four-dimensional ancient κ-solution $(M^4, g_{ij}(t)), -\infty < t \leq 0$, with restricted isotropic curvature pinching for some $\kappa > 0$. Then*

(i) *for every $x, y \in M^4$ and $t \in (-\infty, 0]$, there holds*

$$R(x,t) \leq R(y,t) \cdot \omega(R(y,t)d_t^2(x,y));$$

(ii) *for all $x \in M^4$ and $t \in (-\infty, 0]$, there hold*

$$|\nabla R|(x,t) \leq \eta R^{\frac{3}{2}}(x,t) \ and \ \left|\frac{\partial R}{\partial t}\right|(x,t) \leq \eta R^2(x,t).$$

The following result generalizes Theorem 11.7 of Perelman [**31**] to four-dimension.

Corollary 3.7. *The set of four-dimensional ancient κ-solutions with restricted isotropic curvature pinching and **positive** curvature operator is precompact modulo scaling in the sense that for any sequence of such solutions and marked points $(x_k, 0)$ with $R(x_k, 0) = 1$, we can extract a C_{loc}^∞ converging subsequence, and the limit is also an ancient κ_0-solution with restricted isotropic curvature pinching.*

Proof. Consider any sequence of four-dimensional ancient κ-solutions with restricted isotropic curvature pinching and positive curvature operator and marked points $(x_k, 0)$ with $R(x_k, 0) = 1$. By Proposition 3.6 (i), Li-Yau-Hamilton inequality [**18**] and Hamilton's compactness theorem (Theorem 16.1 of [**20**]), we can extract a C_{loc}^∞ converging subsequence such that the limit $(\overline{M}^4, \overline{g}_{ij}(t))$ is an ancient solution to the Ricci flow and satisfies the restricted isotropic curvature pinching condition (2.4), and is κ-noncollapsed for all scales. Moreover, the limit still satisfies the Li-Yau-Hamilton inequality and assertions (i) and (ii) of Proposition 3.6. To show the limit is an ancient κ-solution, it remains to show the limit has bounded curvature at the time $t = 0$.

By the virtue of Lemma 3.2, we may assume the limit has positive curvature operator everywhere. We now argue by contradiction. Suppose the curvature of the limit (at $t = 0$) $(\bar{M}^4, \bar{g}_{ij}(0))$ is unbounded, then there is a sequence of points P_l divergent to infinity at the time $t = 0$ with the scalar curvature $R(P_l, 0) \to +\infty$. By Hamilton's compactness theorem (Theorem 16.1 of [**20**]) and the estimates in assertions (i) and (ii) of Proposition 3.6, we know that a subsequence of the rescaled solutions $(\overline{M}^4, \bar{R}(P_l, 0)\bar{g}_{ij}(\cdot, \frac{t}{R(P_l,0)}), P_l)$ converges in C^∞_{loc} to a smooth nonflat limit. And by Lemma 3.1, the limit must be the round neck $\mathbb{R} \times \mathbb{S}^3$. This contradicts Proposition 2.2.

Therefore we have proved the corollary.   q.e.d.

3.4. Canonical neighborhood structures. We now examine the structures of four-dimensional nonflat ancient κ-solutions with restricted isotropic curvature pinching. As before by Lemma 3.2, we have seen that a four-dimensional nonflat ancient κ-solution with restricted isotropic curvature pinching, whose curvature operator has a nontrivial null vector somewhere at some time, must be a metric quotient of the round cylinder $\mathbb{R} \times \mathbb{S}^3$. So we only need to consider the ancient κ-solutions with **positive** curvature operator. The following theorem gives their canonical neighborhood structures. The analogous result in the three-dimensional case was given by Perelman in Section 1.5 of [**32**].

Theorem 3.8. *For every $\varepsilon > 0$ one can find positive constants $C_1 = C_1(\varepsilon)$, $C_2 = C_2(\varepsilon)$ such that for each point (x, t) in every four-dimensional ancient κ-solution (for some $\kappa > 0$) with restricted isotropic curvature pinching and with **positive** curvature operator, there is a radius r, $0 < r < C_1(R(x, t))^{-\frac{1}{2}}$, so that some open neighborhood $B_t(x, r) \subset B \subset B_t(x, 2r)$ falls into one of the following three categories:*

(a) *B is an **evolving ε-neck** (in the sense that it is the time slice at time t of the parabolic region $\{(x', t') | x' \in B, t' \in [t - \varepsilon^{-2}R(x, t)^{-1}, t]\}$ which is, after scaling with factor $R(x, t)$ and shifting the time t to 0, ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the subset $(\mathbb{I} \times \mathbb{S}^3) \times [-\varepsilon^{-2}, 0]$ of the evolving round cylinder $\mathbb{R} \times \mathbb{S}^3$, having scalar curvature one and length $2\varepsilon^{-1}$ to $\mathbb{I}$ at time zero), or*

(b) *B is an **evolving ε-cap** (in the sense that it is the time slice at the time t of an evolving metric on open $\mathbb{B}^4$ or $\mathbb{RP}^4 \setminus \overline{\mathbb{B}^4}$ such that the region outside some suitable compact subset of $\mathbb{B}^4$ or $\mathbb{RP}^4 \setminus \overline{\mathbb{B}^4}$ is an evolving ε-neck), or*

(c) *B is a compact manifold (without boundary) with positive curvature operator (thus it is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$); furthermore, the scalar curvature of the ancient κ-solution in B at time t is between $C_2^{-1}R(x, t)$ and $C_2 R(x, t)$, and the volume of*

B in case (a) *and case* (b) *satisfies*

$$(C_2 R(x,t))^{-2} \leq \mathrm{Vol}_t(B).$$

Proof. If the nonflat ancient κ-solution is noncompact, the conclusions follow immediately from (the proof of) Proposition 3.4. We thus assume the nonflat ancient κ-solution is compact. By Theorem 3.5 we see that such an ancient κ-solution is κ_0-noncollapsed for all scales for some universal positive constant κ_0.

We argue by contradiction. Suppose for some $\varepsilon > 0$, there exists a sequence of compact ancient κ_0-solutions (M_k^4, g_k) with restricted isotropic curvature pinching and with positive curvature operator, a sequence of points $x_k \in M_k^4$, and sequences of positive constants C_{1k} with $C_{1k} \to +\infty$ as $k \to +\infty$ and $C_{2k} = \omega(4C_{1k}^2)$ with the function ω given in Proposition 3.6 such that at time t, for every radius r, $0 < r < C_{1k}R(x_k,t)^{-\frac{1}{2}}$, any open neighborhood B with $B_t(x_k, r) \subset B \subset B_t(x_k, 2r)$ can not fall into any one of the three categories (a), (b) and (c). Clearly, the diameter of each M_k^4 at time t is at least $C_{1k}R(x_k,t)^{-\frac{1}{2}}$; otherwise one can choose suitable $r \in (0, C_{1k}R(x_k,t)^{-\frac{1}{2}})$ and $B = M_k^4$, which falls into category (c), so that the scalar curvature in B at t is between $C_{2k}^{-1}R(x_k,t)$ and $C_{2k}R(x_k,t)$ by using Proposition 3.6 (i). Now by scaling the ancient κ_0-solutions along the points (x_k,t) with factors $R(x_k,t)$ and shifting the time t to 0, it follows from Corollary 3.7 that a subsequence of these rescaled ancient κ_0-solutions converges in C_{loc}^∞ topology to a noncompact nonflat ancient κ_0-solution with restricted isotropic curvature pinching.

If the noncompact limit has a nontrivial null curvature eigenvector somewhere, then by Lemma 3.2 we conclude that the limit is round cylinder $\mathbb{R} \times \mathbb{S}^3$ or a metric quotient $\mathbb{R} \times \mathbb{S}^3/\Gamma$. By the same reason as in the proof of Theorem 3.5, the projection Γ_1 of Γ to the factor $\mathbb{R}$ is an isometric subgroup of $\mathbb{R}$. Since the limit $\mathbb{R} \times \mathbb{S}^3/\Gamma$ is noncompact, Γ_1 must be $\{1\}$ or $\mathbb{Z}_2$. Thus we have a Γ-invariant cross-sphere $\mathbb{S}^3$ in $\mathbb{R} \times \mathbb{S}^3/\Gamma$, and Γ acts on it without fixed points. Denote this cross sphere by $\{0\} \times \mathbb{S}^3$. Since each (M_k^4, g_k) is compact and has positive curvature operator, we know from [16] that each M_k^4 is diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$. Then by the proof of Theorem 3.5 and applying theorem C 4.1 of [21], we conclude that the limit is $\mathbb{R} \times \mathbb{S}^3$ or $\mathbb{R} \times \mathbb{S}^3/\Gamma$ with $\Gamma = \mathbb{Z}_2$. If $\Gamma = \mathbb{Z}_2$, we claim that Γ must act on $\mathbb{R} \times \mathbb{S}^3/\Gamma$ by flipping both $\mathbb{R}$ and $\mathbb{S}^3$.

Indeed, as shown before, $\Gamma_1 = \{1\}$ or $\mathbb{Z}_2$. If $\Gamma_1 = \{1\}$, then $\mathbb{R} \times \mathbb{S}^3/\Gamma = \mathbb{R} \times \mathbb{RP}^3$. Let Γ^+ be the normal subgroup of Γ preserving the orientation of the cylinder, and $\pi_1(M_k^4)^+$ be the normal subgroup of $\pi_1(M_k^4)$ preserving the orientation of the universal cover of M_k^4. Since the manifold M_k^4 is diffeomorphic to $\mathbb{RP}^4$, this induces an absurd commutative diagram:

$$
\begin{array}{ccccccccc}
 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & & 0 & & \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \Gamma^+ & \longrightarrow & \Gamma & \longrightarrow & \Gamma/\Gamma^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \wr & & \downarrow & & \\
0 & \longrightarrow & \pi_1(M_k^4)^+ & \longrightarrow & \pi_1(M_k^4) & \longrightarrow & \pi_1(M_k^4)/\pi_1(M_k^4)^+ & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
 & & 0 & & \mathbb{Z}_2 & & \mathbb{Z}_2 & &
\end{array}
$$

where the vertical morphisms are induced by the inclusion $\mathbb{R} \times \mathbb{S}^3/\Gamma \subset M$. Therefore $\Gamma_1 = \mathbb{Z}_2$. Denote by σ_1 the isometry of $\mathbb{R} \times \mathbb{S}^3$ acting by flipping both $\mathbb{R}$ and $\mathbb{S}^3$ around $\{0\} \times S^3$. Clearly, for any $\sigma \in \Gamma$ with $\sigma \neq 1$, $\sigma \circ \sigma_1$ is an isometry of $\mathbb{R} \times \mathbb{S}^3$ whose projection on the factor $\mathbb{R}$ is the identity map. Then $\sigma \circ \sigma_1$ is only a rotation of the factor $\mathbb{S}^3$ in $\mathbb{R} \times \mathbb{S}^3$. Note that $\sigma \circ \sigma_1 \mid_{\{0\} \times \mathbb{S}^3}$ is an identity. We conclude that $\sigma = \sigma_1$ and the claim holds.

When the limit is the round cylinder $\mathbb{R} \times \mathbb{S}^3$, a suitable neighborhood B (for suitable r) of x_k would fall into category (a) for sufficiently large k; while when the limit is the $\mathbb{Z}_2$ quotient of the round cylinder $\mathbb{R} \times \mathbb{S}^3$ with the antipodal map flipping both $\mathbb{S}^3$ and $\mathbb{R}$, a suitable neighborhood B (for suitable r) of x_k would fall into category (b) (over $\mathbb{RP}^4 \setminus \overline{\mathbb{B}^4}$) or into category (a) for sufficiently large k. This is a contradiction.

If the noncompact limit has positive curvature operator everywhere, then by Proposition 3.4, a suitable neighborhood B (for suitable r) of x_k would fall into category (b) (over $\mathbb{B}^4$) for sufficiently large k. We also get a contradiction.

Finally, the statements on the curvature estimate and volume estimate for the neighborhood B follow directly from Theorem 3.6 and Proposition 3.4. Therefore we have proved the theorem.   q.e.d.

4. The Structure of Solutions at the Singular Time

Let $(M^4, g_{ij}(x))$ be a four-dimensional compact Riemannian manifold with positive isotropic curvature and let $g_{ij}(x,t)$, $x \in M^4$ and $t \in [0, T)$, be a maximal solution to the Ricci flow (1.1) with $g_{ij}(x,0) = g_{ij}(x)$ on M^4. Since the initial metric $g_{ij}(x)$ has positive scalar curvature, it is easy to see that the maximal time T must be finite and the curvature tensor becomes unbounded as $t \to T$. According to Perelman's noncollapsing theorem I (Theorem 4.1 of [**31**]), the solution $g_{ij}(x,t)$ is κ-noncollapsed on the scale $\sqrt{T}$ for all $t \in [0, T)$ for some $\kappa > 0$. Now let us take a sequence of times $t_k \to T$, and a sequence of points $p_k \in M^4$ such that for some positive constant C, $|R_m|(x,t) \leq CQ_k$ with $Q_k = |Rm(p_k, t_k)|$ whenever $x \in M^4$ and $t \in [0, t_k]$, called a sequence of (almost) **maximal points**. Then by Hamilton's compactness theorem

[**19**], a sequence of the scalings of the solution $g_{ij}(x,t)$ along the points p_k with factors Q_k converges to a complete ancient κ-solution with restricted isotropic curvature pinching. This says that, for any $\varepsilon > 0$, there exists a positive number k_0 such that as $k \geq k_0$, the solution in the parabolic region $\{(x,t) \in M^4 \times [0,T) \mid d_{t_k}^2(x, x_k) < \varepsilon^{-2}Q_k^{-1}, t_k - \varepsilon^{-2}Q_k^{-1} < t \leq t_k\}$ is, after scaling with the factor Q_k, ε-close (in $C^{[\varepsilon^{-1}]}$-topology) to the corresponding subset of the ancient κ-solution with restricted isotropic curvature pinching.

Let us describe the structure of any ancient κ-solution (with restricted isotropic curvature pinching). If the curvature operator is positive everywhere, then each point of the ancient κ-solution has a canonical neighborhood described in Theorem 3.8. Meanwhile if the curvature operator has a nontrivial null eigenvector somewhere, then by Hamilton's strong maximum principle and the pinching condition (2.4) the ancient κ-solution is isometric to $\mathbb{R} \times \mathbb{S}^3/\Gamma$, a metric quotient of the round cylinder $\mathbb{R} \times \mathbb{S}^3$. Since it is κ-noncollapsed for all scales, the metric quotient $\mathbb{R} \times \mathbb{S}^3/\Gamma$ can not be compact. Suppose we make an additional assumption that the compact four manifold M^4 has no essential incompressible space form. Then by the proofs of Theorems 3.5 and 3.8 and applying Theorem C 4.1 of [**21**], we have $\Gamma = \{1\}$, or $\Gamma = \mathbb{Z}_2$ acting antipodally on $\mathbb{S}^3$ and by reflection on $\mathbb{R}$. Thus in both cases, each point of the ancient κ solution has also a canonical neighborhood described in Theorem 3.8.

Hence we see that each such (almost) maximal point (x_k, t_k) has a canonical neighborhood which is either an evolving ε-neck or an evolving ε-cap, or a compact manifold (without boundary) with positive curvature operator. This gives the structure of the singularities coming from a sequence of (almost) maximal points (x_k, t_k). However, this argument does not work for the singularities coming from a sequence of points (y_k, τ_k) with $\tau_k \to T$ and $|Rm(y_k, \tau_k)| \to +\infty$ when $|Rm(y_k, \tau_k)|$ is not comparable with the maximum of the curvature at the time τ_k, since we can not take a limit directly.

We now follow a refined rescaling argument of Perelman (Theorem 12.1 of [**31**], see also Theorem 51.3 in [**26**] and Theorem 7.1.1 in [**3**] for the details) to obtain a uniform canonical neighborhood structure theorem for four-dimensional solutions at any point where its curvature is suitably large.

Theorem 4.1. *Given $\varepsilon > 0$, $\kappa > 0$, $0 < \theta, \rho, \Lambda, P < +\infty$, one can find $r_0 > 0$ with the following property. If $g_{ij}(x,t)$, $t \in [0,T)$ with $T > 1$, is a solution to the Ricci flow on a four-dimensional compact manifold M^4 with no essential incompressible space form, which has positive isotropic curvature, is κ-noncollapsed on the scales $\leq \theta$ and satisfies (2.1), (2.2) and (2.3) in Lemma 2.1, then for any point (x_0, t_0) with $t_0 \geq 1$ and $Q = R(x_0, t_0) \geq r_0^{-2}$, the solution in the parabolic*

region $\{(x, t) \in M^4 \times [0, T)| d_{t_0}^2(x, x_0) < \varepsilon^{-2}Q^{-1}, t_0 - \varepsilon^{-2}Q^{-1} < t \le t_0\}$ is, after scaling by the factor Q, ε-close (in $C^{[\varepsilon^{-1}]}$-topology) to the corresponding subset of some ancient κ-solution with restricted isotropic curvature pinching.

Consequently each point (x_0, t_0), with $t_0 \ge 1$ and $Q = R(x_0, t_0) \ge r_0^{-2}$, satisfies the gradient estimates

$$(4.1) \quad |\nabla R(x_0, t_0)| < 2\eta R^{\frac{3}{2}}(x_0, t_0) \ and \ \left|\frac{\partial}{\partial t} R(x_0, t_0)\right| < 2\eta R^2(x_0, t_0),$$

and has a canonical neighborhood B with $B_{t_0}(x_0, r) \subset B \subset B_{t_0}(x_0, 2r)$ for some $0 < r < C_1(\varepsilon)(R(x_0, t_0))^{-\frac{1}{2}}$, which is either an evolving ε-neck, or an evolving ε-cap, or a compact four-manifold with positive curvature operator. Here η is the universal constant in Proposition 3.6 and $C_1(\varepsilon)$ is the positive constant in Theorem 3.8.

Proof. The detail exposition of Perelman's refined rescaling argument have been given in Theorem 51.3 of [26] and Theorem 7.1.1 of [3] for three-manifolds. We now adapt that to the four-dimensional Ricci flow.

Let $C(\varepsilon)$ be a positive constant depending only on ε such that $C(\varepsilon) \to +\infty$ as $\varepsilon \to 0^+$. It suffices to prove that there exists $r_0 > 0$ such that for any point (x_0, t_0) with $t_0 \ge 1$ and $Q = R(x_0, t_0) \ge r_0^{-2}$, the solution in the parabolic region $\{(x, t) \in M^4 \times [0, T) \mid d_{t_0}^2(x, x_0) < C(\varepsilon)Q^{-1}, t_0 - C(\varepsilon)Q^{-1} < t \le t_0\}$ is, after scaling by the factor Q, ε-close to the corresponding subset of some ancient κ-solution with restricted isotropic curvature pinching. The constant $C(\varepsilon)$ will be determined later.

We argue by contradiction. Suppose for some $\varepsilon > 0$, $\kappa > 0$, $0 < \theta, \rho, \Lambda, P < +\infty$, there exists a sequence of solutions $(M_k^4, g_{ij}^{(k)}(\cdot, t))$ to the Ricci flow on compact four-manifolds with no essential incompressible space form, having positive isotropic curvature and satisfying (2.1), (2.2) and (2.3), defined on the time interval $[0, T_k)$ with $T_k > 1$, and a sequence of positive numbers $r_k \to 0$ such that each solution $(M_k^4, g_{ij}^{(k)}(\cdot, t))$ is κ-noncollapsed on the scales $\le \theta$; but there exists a sequence of points $x_k \in M_k^4$ and times $t_k \ge 1$ with $Q_k = R_k(x_k, t_k) \ge r_k^{-2}$ such that the solution in the parabolic region $\{(x, t) \in M_k^4 \times [0, T_k) \mid d_{t_k}^2(x, x_k) < C(\varepsilon)Q_k^{-1}, t_k - C(\varepsilon)Q_k^{-1} < t \le t_k\}$ is not, after scaling by the factor Q_k, ε-close to the corresponding subset of any ancient κ-solution with restricted isotropic curvature pinching, where R_k denotes the scalar curvature of $(M_k^4, g_{ij}^{(k)}(\cdot, t))$. For each solution $(M_k^4, g_{ij}^{(k)}(\cdot, t))$, we may adjust the point (x_k, t_k) with $t_k \ge \frac{1}{2}$ and with $Q_k = R_k(x_k, t_k)$ as large as possible so that the conclusion of the theorem fails at (x_k, t_k), but holds for any $(x, t) \in M_k^4 \times [t_k - H_kQ_k^{-1}, t_k]$

satisfying $R(x,t) \geq 2Q_k$, where $H_k = \frac{1}{4}r_k^{-2} \to +\infty$ as $k \to +\infty$. Indeed, suppose not; by setting $(x_k^{(1)}, t_k^{(1)}) = (x_k, t_k)$, we can inductively choose $(x_k^{(\ell)}, t_k^{(\ell)}) \in M_k^4 \times [t_k^{(\ell-1)} - H_k(R_k(x_k^{(\ell-1)}, t_k^{(\ell-1)}))^{-1}, t_k^{(\ell-1)}]$ satisfying $R_k(x_k^{(\ell)}, t_k^{(\ell)}) \geq 2R_k(x_k^{(\ell-1)}, t_k^{(\ell-1)})$, but the conclusion of the theorem fails at $(x_k^{(\ell)}, t_k^{(\ell)})$ for each $\ell = 2, 3, \ldots$. Since the solution is smooth and

$$
\begin{aligned}
R_k(x_k^{(\ell)}, t_k^{(\ell)}) &\geq 2R_k(x_k^{(\ell-1)}, t_k^{(\ell-1)}) \\
&\geq 2^{\ell-1} R_k(x_k, t_k),
\end{aligned}
$$

$$
\begin{aligned}
t_k^{(\ell)} &\geq t_k^{(\ell-1)} - H_k(R_k(x_k^{(\ell-1)}, t_k^{(\ell-1)}))^{-1} \\
&\geq t_k - H_k \sum_{i=1}^{\ell-1} (2^{i-1} R_k(x_k, t_k))^{-1} \\
&\geq t_k - 2H_k(R(x_k, t_k))^{-1} \\
&\geq \frac{1}{2},
\end{aligned}
$$

the above choosing process must terminate in finite step and the last element fits.

Let $(M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, t), x_k)$ be the rescaled solutions obtained by rescaling the manifolds $(M_k^4, g_{ij}^{(k)}(\cdot, t))$ with factors $Q_k = R_k(x_k, t_k)$ and shifting the time t_k to 0. Denote by $\widetilde{R}_k$ the rescaled scalar curvature. We will show that a subsequence of the rescaled solutions $(M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, t), x_k)$ converges to an ancient κ-solution with restricted isotropic curvature pinching, which is a contradiction. In the following we divide the argument into four steps.

Step 1.   We want to prove a local curvature estimate in the following assertion. This is the four-dimensional version of the claim 1 of Perelman in the proof of Theorem 12.1 of [**31**].

Claim. For each $(\overline{x}, \overline{t})$ with $t_k - \frac{H_k}{2}Q_k^{-1} < \overline{t} \leq t_k$, we have $R_k(x, t) \leq 4\overline{Q}_k$ whenever $\overline{t} - c\overline{Q}_k^{-1} \leq t \leq \overline{t}$ and $d_{\overline{t}}^2(x, \overline{x}) \leq c\overline{Q}_k^{-1}$, where $\overline{Q}_k = Q_k + R_k(\overline{x}, \overline{t})$ and $c > 0$ is a small universal constant.

To prove this, we consider any point $(x, t) \in B_{\overline{t}}(\overline{x}, (c\overline{Q}_k^{-1})^{\frac{1}{2}}) \times [\overline{t} - c\overline{Q}_k^{-1}, \overline{t}]$ with $c > 0$ to be determined. Clearly, we may assume $R_k(x, t) > 2Q_k$. Let us draw a space time curve γ that goes from (x, t) to $(\overline{x}, t)$ along a minimizing geodesic (with respect to the metric $g_{ij}^{(k)}(\cdot, t)$) and goes straight from $(\overline{x}, t)$ to $(\overline{x}, \overline{t})$. If there exists a point on γ with the scalar curvature $2Q_k$, we choose p to be the nearest such point to (x, t); otherwise, we choose $p = (\overline{x}, \overline{t})$. On the segment of γ from (x, t) to p, the scalar curvature is not less than $2Q_k$. According to the choice of the point (x_k, t_k), the solution along the segment is ε-close to that

of some ancient κ-solutions with restricted isotropic curvature pinching. Of course we may assume $\varepsilon > 0$ is very small. It follows from Proposition 3.6 (ii) that

$$\left|\nabla(R_k^{-\frac{1}{2}})\right| \leq 2\eta \text{ and } \left|\frac{\partial}{\partial t}(R_k^{-1})\right| \leq 2\eta$$

on the segment for some universal constant $\eta > 0$. Then by choosing $c > 0$ (depending only on η) small enough we get the desired curvature bound by integrating the above derivative estimate along the segment. This proves the assertion.

Step 2.   We next want to show that the curvatures of the rescaled solutions $\widetilde{g}_{ij}^{(k)}(\cdot, t)$ at the new time $t = 0$ (i.e., the original time t_k) stay uniformly bounded at bounded distances from x_k. This is a weaker and four-dimensional version of the claim 2 of Perelman in the proof of Theorem 12.1 of [31]. We remark that the first detail exposition of this part for three-dimension appeared (in June 2003) in the first version of Kleiner-Lott [26].

For all $\sigma \geq 0$, set

$$M(\sigma) = \sup\{\widetilde{R}_k(x, 0) \mid k \geq 1, x \in M_k^4 \text{ with } d_0(x, x_k) \leq \sigma\}$$

and

$$\sigma_0 = \sup\{\sigma \geq 0 \mid M(\sigma) < +\infty\}.$$

Note that $\sigma_0 > 0$ by Step 1. By assumptions (2.1) and (2.2), it suffices to show $\sigma_0 = +\infty$. We now argue by contradiction to show $\sigma_0 = +\infty$. Suppose not; then after passing to a subsequence, we can find a sequence of points $y_k \in M_k^4$ so that $d_0(y_k, x_k) \to \sigma_0 < +\infty$ and $\widetilde{R}_k(y_k, 0) \to +\infty$ as $k \to +\infty$. Let $\gamma_k(\subset M_k^4)$ be a minimizing geodesic segment from x_k to y_k, and choose $z_k \in \gamma_k$ to be the point on γ_k closest to y_k at which $\widetilde{R}_k(z_k, 0) = 4$. Denote by β_k the subsegment of γ_k running from y_k to z_k. It follows from the claim in Step 1 that the length of β_k is uniformly bounded away from zero for all k. By assumptions (2.1) and (2.2), we have a uniform curvature bound on the open balls $B_0(x_k, \sigma) \subset (M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, 0))$ for each fixed $\sigma < \sigma_0$. Note that the κ-noncollapsing assumption implies the uniform injectivity radius bound for $(M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, 0))$ at the marked points x_k. Then by the virtue of Hamilton's compactness theorem 16.1 in [20] (see [3] for details on generalizing Hamilton's compactness theorem to finite balls) and the claim in Step 1, we can extract a subsequence of the marked $(B_0(x_k, \sigma_0), \widetilde{g}_{ij}^{(k)}(\cdot, 0), x_k)$ which converges in C_{loc}^∞ topology to a marked (noncomplete) manifold $(B_\infty, \widetilde{g}_{ij}^\infty, x_\infty)$, so that the segments γ_k converge to a geodesic segment (missing an endpoint) $\gamma_\infty \subset B_\infty$ emanating from x_∞, and β_k converge to a subsegment β_∞ of γ_∞. Let $\overline{B}_\infty$ denote the completion of $(B_\infty, \widetilde{g}_{ij}^{(\infty)})$, and $y_\infty \in \overline{B}_\infty$ the limit point of γ_∞.

Denote by $\widetilde{R}_\infty$ the scalar curvature of $(B_\infty, \widetilde{g}_{ij}^{(\infty)})$. Since the rescaled scalar curvatures of $\widetilde{R}_k$ along β_k are at least 3, it follows from the choice of the points (x_k, t_k) that for any $q_0 \in \beta_\infty$, the manifold $(B_\infty, \widetilde{g}_{ij}^{(\infty)})$ in $\{q \in B_\infty | dist^2_{\widetilde{g}_{ij}^{(\infty)}}(q, q_0) < C(\varepsilon)(\widetilde{R}_\infty(q_0))^{-1}\}$ is 2ε-close to the corresponding subset of (a time slice of) some ancient κ-solution with restricted isotropic curvature pinching. From the argument in the second paragraph of this section, we know that such an ancient κ-solution with restricted isotropic curvature pinching at each point (x, t) has a radius r, $0 < r < C_1(2\varepsilon)R(x, t)^{-\frac{1}{2}}$, such that its canonical neighborhood B with $B_t(x, r) \subset B \subset B_t(x, 2r)$, is either an evolving 2ε-neck, or an evolving 2ε-cap, or a compact manifold (without boundary) with positive curvature operator; moreover the scalar curvature on the ball is between $C_2(2\varepsilon)^{-1}R(x, t)$ and $C_2(2\varepsilon)R(x, t)$, where $C_1(2\varepsilon)$ and $C_2(2\varepsilon)$ are the positive constants in Theorem 3.8. We now choose $C(\varepsilon) = \max\{2C_1(2\varepsilon)^2, \varepsilon^{-2}\}$. By the local curvature estimate in Step 1, we see that the scalar curvature $\widetilde{R}_\infty$ becomes unbounded along γ_∞ going to y_∞. This implies that the canonical neighborhood around q_0 can not be a compact manifold (without boundary) with positive curvature operator. Note that γ_∞ is shortest since it is the limit of a sequence of shortest geodesics. Without loss of generality, we may assume ε is suitably small. This implies that as q_0 is sufficiently close to y_∞, the canonical neighborhood around q_0 can not be a 2ε-cap. Thus we conclude that each $q_0 \in \gamma_\infty$, which is sufficiently close to y_∞, is the center of a 2ε-neck.

Denote by

$$U = \bigcup_{q_0 \in \gamma_\infty} B(q_0, 24\pi(\widetilde{R}_\infty(q_0))^{-\frac{1}{2}}) \quad (\subset (B_\infty, \widetilde{g}_{ij}^{(\infty)}))$$

where $B(q_0, 24\pi(\widetilde{R}_\infty(q_0))^{-\frac{1}{2}})$ is the ball centered at q_0 of radius $24\pi(\widetilde{R}_\infty(q_0))^{-\frac{1}{2}}$.

Clearly, it follows from assumptions (2.1), (2.2) and (2.3) that U has nonnegative curvature operator. Since the metric $\widetilde{g}_{ij}^{(\infty)}$ is cylindrical at any point $q_0 \in \gamma_\infty$ which is sufficiently close to y_∞, we see that the metric space $\overline{U} = U \cup \{y_\infty\}$ by adding the point y_∞, is locally complete and strictly intrinsic near y_∞. Here strictly intrinsic means that the distance between any two points can be realized by shortest geodesics. Furthermore y_∞ cannot be an interior point of any geodesic segment in $\overline{U}$. This implies that the curvature of $\overline{U}$ at y_∞ is nonnegative in Alexandrov sense. Note that for any very small radius σ, the geodesic sphere $\partial B(y_\infty, \sigma)$ is an almost round sphere of radius $\leq 3\varepsilon\sigma\pi$. By [**1**] or [**6**] we have a four-dimensional tangent cone at y_∞ with aperture $\leq 20\varepsilon$. Moreover, by [**1**] or [**6**], any four-dimensional tangent cone $C_{y_\infty}\overline{U}$ at

y_∞ must be a metric cone. For each tangent cone, pick $z \in C_{y_\infty} \overline{U}$ such that the distance between the vertex y_∞ and z is one. Then the ball $B(z, \frac{1}{2}) \subset C_{y_\infty} \overline{U}$ is the Gromov-Hausdorff limit of the scalings of a sequence of balls $B_0(z_k, s_k) \subset (M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, 0))$ by some factors a_k, where $s_k \to 0^+$. Since the tangent cone is four-dimensional and has aperture $\leq 20\varepsilon$, the factors a_k must be comparable with $\tilde{R}_k(z_k, 0)$. By using the local curvature estimate in Step 1, we actually have the convergence in the C_{loc}^∞ topology for the solutions $\widetilde{g}_{ij}^{(k)}(\cdot, t)$ over the balls $B_0(z_k, s_k)$ and over some time interval $t \in [-\delta, 0]$ for some sufficiently small $\delta > 0$. The limiting $B(z, \frac{1}{2}) \subset C_{y_\infty} \overline{U}$ is a piece of the nonnegative (operator) curved and nonflat metric cone. On the other hand, since the radial direction of the cone is flat, by Hamilton's strong maximum principle [**16**] and the pinching condition (2.4) as in the proof of Lemma 3.2, the limiting $B(z, \frac{1}{2})$ would be a piece of $\mathbb{R} \times \mathbb{S}^3$ or $\mathbb{R} \times \mathbb{S}^3/\Gamma$ (a metric quotient). This is a contradiction. So we have proved that the curvatures of the rescaled metrics $\widetilde{g}_{ij}^{(k)}(\cdot, 0)$ stay uniformly bounded at bounded distances from x_k.

By the local curvature estimate in Step 1, we can locally extend the above curvature control backward in time a little. Then by the κ-noncollapsing assumption and Shi's derivative estimates [**36**], we can take a C_{loc}^∞ limit from the sequence of marked rescaled solutions $(M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, t), x_k)$. The limit, denoted by $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t), x_\infty)$, is κ-noncollapsing on all scales, is defined on a space-time open subset of $M_\infty^4 \times (-\infty, 0]$ containing the time slice $M_\infty^4 \times \{0\}$, and satisfies the restricted isotropic curvature pinching condition (2.4) by the assumptions (2.1), (2.2) and (2.3).

Step 3. We further claim that the limit $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t))$ at the time slice $t = 0$ has bounded curvature.

We have known that the curvature operator of the limit $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t))$ is nonnegative everywhere. If the curvature operator has a nontrivial null eigenvector somewhere, we can argue as in the proof of Lemma 3.2 by using Hamilton's strong maximum principle [**16**] and the restricted isotropic curvature pinching condition (2.4) to deduce that the universal cover of the limit is isometric to the standard $\mathbb{R} \times \mathbb{S}^3$. Thus the curvature of the limit is bounded in this case.

Assume that the curvature operator of the limit $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t))$ at the time slice $t = 0$ is positive everywhere. Suppose there exists a sequence of points $p_j \in M_\infty^4$ such that their scalar curvatures $\widetilde{R}_\infty(p_j, 0) \to +\infty$ as $j \to +\infty$. By the local curvature estimate in Step 1 and the assertion of the above Step 2 (for the marked points p_j) as well as the κ-noncollapsed assumption, a subsequence of the rescaled and marked

manifolds $(M_\infty^4, \widetilde{R}_\infty(p_j, 0)\widetilde{g}_{ij}^{(\infty)}(\cdot), p_j)$ converges in C_{loc}^∞ topology to a smooth nonflat limit Y. Then by Lemma 3.1 we conclude that Y is isometric to $\mathbb{R} \times \mathbb{S}^3$ with the standard metric. This contradicts Proposition 2.2. So the curvature of the limit $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t))$ at the time slice $t = 0$ must be bounded.

Step 4. Finally we want to extend the limit backward in time to $-\infty$.

By the local curvature estimate in Step 1, we now know that the limit $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t))$ is defined on $[-a, 0]$ for some $a > 0$.

Denote by

$$t' = \inf\{\tilde{t} \mid \text{we can take a smooth limit on } (\tilde{t}, 0]$$
$$\text{(with bounded curvature at each time slice)}$$
$$\text{from a subsequence of the rescaled solutions } \tilde{g}_k\}.$$

We first claim that there is a subsequence of the rescaled solutions $\tilde{g}_k$ which converges in C_{loc}^∞ topology to a smooth limit $(M_\infty, \tilde{g}_\infty(\cdot, t))$ on the maximal time interval $(t', 0]$.

Indeed, let t_k be a sequence of negative numbers such that $t_k \to t'$ and there exist smooth limits $(M_\infty, \tilde{g}_\infty^k(\cdot, t))$ defined on $(t_k, 0]$. For each k, the limit has nonnegative and bounded curvature operator at each time slice. Moreover, by the claim in Step 1, the limit has bounded curvature on each subinterval $[-b, 0] \subset (t_k, 0]$. Denote by $\tilde{Q}$ the scalar curvature upper bound of the limit at time zero (where $\tilde{Q}$ is the same for all k). Then we can apply Li-Yau-Hamilton inequality [**18**] to get

$$\tilde{R}_\infty^k(x, t) \leq \tilde{Q}\left(\frac{-t_k}{t - t_k}\right),$$

where $\tilde{R}_\infty^k(x, t)$ are the scalar curvatures of the limits $(M_\infty, \tilde{g}_\infty^k(\cdot, t))$. Hence by the definition of convergence and the above curvature estimates, we can find a subsequence of the rescaled solutions $\tilde{g}_k$ which converges in C_{loc}^∞ topology to a smooth limit $(M_\infty, \tilde{g}_\infty(\cdot, t))$ on the maximal time interval $(t', 0]$.

We next claim that $t' = -\infty$.

Suppose not; then the curvature of the limit $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t))$ becomes unbounded as $t \to t' > -\infty$. Since the minimum of the scalar curvature is nondecreasing in time and $\widetilde{R}_\infty(x_\infty, 0) = 1$, we see that there is a $y_\infty \in M_\infty^4$ such that

$$0 < \widetilde{R}_\infty\left(y_\infty, t' + \frac{c}{3}\right) < \frac{3}{2}$$

where $c > 0$ is the universal constant in the assertion of Step 1. By using Step 1 again we see that the limit $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t))$ in a small neighborhood of the point y_∞ at the time slice $t = t' + \frac{c}{3}$ can be extended

backward to the time interval $[t' - \frac{c}{3}, t' + \frac{c}{3}]$. We remark that the distances at the time t and the time 0 are roughly equivalent in the following sense:

$$(4.2) \qquad d_t(x,y) \geq d_0(x,y) \geq d_t(x,y) - \text{const.}$$

for any $x, y \in M_\infty^4$ and $t \in (t', 0]$. Indeed, from the Li-Yau-Hamilton inequality [**18**] we have the estimate

$$\frac{\partial}{\partial t} \widetilde{R}_\infty(x,t) \geq -\widetilde{R}_\infty(x,t) \cdot (t - t')^{-1}, \text{ for } (x,t) \in M_\infty^4 \times (t', 0].$$

If $\widetilde{Q}$ denotes the supermum of the scalar curvature $\widetilde{R}_\infty$ at $t = 0$, then

$$\widetilde{R}_\infty(x,t) \leq \widetilde{Q}\left(\frac{-t'}{t - t'}\right), \text{ on } M_\infty^4 \times (t', 0].$$

By applying Lemma 8.3 (b) of [**31**], we have

$$d_t(x,y) \leq d_0(x,y) + 30(-t')\sqrt{\widetilde{Q}}$$

for any $x, y \in M_\infty^4$ and $t \in (t', 0]$. On the other hand, since the curvature operator of the limit $\widetilde{g}_{ij}^\infty(\cdot, t)$ is nonnegative, we have

$$d_t(x,y) \geq d_0(x,y)$$

for any $x, y \in M_\infty^4$ and $t \in (t', 0]$. Thus we obtain the estimate (4.2).

The estimate (4.2) insures that the limit around the point y_∞ at any time $t \in (t', 0]$ is exactly the original limit around x_∞ at the time $t = 0$. Consider the rescaled sequence of $(M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, t))$ with the marked points replaced by the associated sequence $y_k \to y_\infty$. By applying the same arguments as the above Step 2 and Step 3 to the new marked sequence $(M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, t), y_k)$, we conclude the original limit $(M_\infty^4, \widetilde{g}_{ij}^{(\infty)}(\cdot, t))$ is actually well defined on the time slice $M_\infty^4 \times \{t'\}$ and also has uniformly bounded curvature for all $t \in [t', 0]$. By taking a subsequence from the original subsequence and combining Step 1, we can extend the limit backward to a larger interval $[t'', 0] \supsetneq (t', 0]$. This is a contradiction with the definition of t'.

Therefore we have proved that a subsequence of the rescaled solutions $(M_k^4, \widetilde{g}_{ij}^{(k)}(\cdot, t), x_k)$ converges to an ancient κ-solution with restricted isotropic curvature pinching. This is a contradiction. We finish the proof of the theorem. \hfill q.e.d.

From now on, we always assume that the initial datum is a compact four-manifold M^4 with no essential incompressible space form and with positive isotropic curvature. Let $g_{ij}(x,t)$, $x \in M^4$ and $t \in [0, T)$, be a maximal solution to the Ricci flow with $T < +\infty$. Without loss of generality, after a scaling on the initial metric, we may assume $T > 1$. It was shown in [**21**] that the solution $g_{ij}(x,t)$ remains positive isotropic curvature. By Lemma 2.1, there hold (2.1), (2.2) and (2.3) for some positive constants $0 < \rho, \Lambda, P < +\infty$ (depending only on the

initial datum). And by Perelman's no local collapsed theorem I [**31**] the solution is κ-noncollapsed on the scale $\sqrt{T}$ for some $\kappa > 0$ (depending only on the initial datum). Then for any sufficiently small $\varepsilon > 0$, we can find $r_0 > 0$ with the property described in Theorem 4.1.

Let Ω denote the set of all points in M^4, where curvature stays bounded as $t \to T$. The estimates (4.1) imply that Ω is open and $R(x,t) \to +\infty$ as $t \to T$ for each $x \in M^4 \backslash \Omega$. If Ω is empty, then the solution becomes extinct at time T and the manifold is either diffeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$, or entirely covered by evolving ε-necks or evolving ε-caps shortly before the maximal time T, so M^4 is diffeomorphic to $\mathbb{S}^4$, or $\mathbb{RP}^4$, or $\mathbb{RP}^4 \# \mathbb{RP}^4$ or $\mathbb{S}^3 \times \mathbb{S}^1$, or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$. The reason is as follows. We only need to consider the situation that the manifold M^4 is entirely covered by evolving ε-necks and evolving ε-caps shortly before the maximal time T. If M^4 contains a cap C, then there is a cap or a neck adjacent to the neck-like end of C. The former case implies that M^4 is diffeomorphic to $\mathbb{S}^4$, $\mathbb{RP}^4$, or $\mathbb{RP}^4 \# \mathbb{RP}^4$. In the latter case, we get a new, longer cap and continue the procedure. Finally, we must end up with a cap, producing a $\mathbb{S}^4$, $\mathbb{RP}^4$, or $\mathbb{RP}^4 \# \mathbb{RP}^4$. If M^4 contains no caps, we start with a neck N, consider the other necks adjacent to the boundary of N, this gives a longer neck and we continue the procedure. After a finite number of steps, the neck must repeat itself. By considering the orientation of M^4, we conclude that M^4 is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$.

We can now assume that Ω is not empty. By using the local derivative estimates of Shi [**36**] (or see [**20**]), we see that as $t \to T$, the solution $g_{ij}(\cdot, t)$ has a smooth limit $\overline{g}_{ij}(\cdot)$ on Ω. Let $\overline{R}(x)$ denote the scalar curvature of $\overline{g}_{ij}$. By the positive isotropic curvature assumption on the initial metric, we know that the metric $\overline{g}_{ij}(\cdot)$ also has positive isotropic curvature; in particular, $\overline{R}(x)$ is positive. For any $\sigma < r_0$, let us consider the set

$$\Omega_\sigma = \{x \in \Omega \mid \overline{R}(x) \leq \sigma^{-2}\}.$$

Note that for any fixed $x \in \partial\Omega$, as $x_j \in \Omega$ and $x_j \to x$ with respect to the initial metric $g_{ij}(\cdot, 0)$, we have $\overline{R}(x_j) \to +\infty$. In fact, if there was a subsequence x_{j_k} so that the limit $\lim_{k \to \infty} \overline{R}(x_{j_k})$ exists and is finite, then it would follow from the gradient estimates (4.1) that $\overline{R}$ is uniformly bounded in some small neighborhood of $x \in \partial\Omega$ (with respect to the induced topology of the initial metric $g_{ij}(\cdot, 0)$); this is a contradiction. From this observation and the compactness of the initial manifold, we see that Ω_σ is compact (with respect to the metric $\overline{g}_{ij}(\cdot)$).

For the further discussion, we follow [**32**] to introduce the following terminologies. Denote by $\mathbb{I}$ a (finite or infinite) interval.

Recall that an ε-**neck** (of radius r) is an open set with a Riemannian metric, which is, after scaling the metric with factor r^{-2}, ε-close (in

$C^{[\varepsilon^{-1}]}$ topology) to the standard neck $\mathbb{S}^3 \times \mathbb{I}$ with the product metric, where $\mathbb{S}^3$ has constant scalar curvature one and $\mathbb{I}$ has length $2\varepsilon^{-1}$. A metric on $\mathbb{S}^3 \times \mathbb{I}$, such that each point is contained in some ε-neck, is called an ε-**tube**, or an ε-**horn**, or a **double** ε-**horn**, if the scalar curvature stays bounded on both ends, or stays bounded on one end and tends to infinity on the other, or tends to infinity on both ends, respectively. A metric on $\mathbb{B}^4$ or $\mathbb{RP}^4 \backslash \overline{\mathbb{B}^4}$, such that each point outside some compact subset is contained in an ε-neck, is called an ε-**cap** or a **capped** ε-**horn**, if the scalar curvature stays bounded or tends to infinity on the end, respectively.

Now take any ε-neck in $(\Omega, \overline{g}_{ij})$ and consider a point x on one of its boundary components. If $x \in \Omega \backslash \Omega_\sigma$, then there is either an ε-cap or an ε-neck, adjacent to the initial ε-neck. In the latter case we can take a point on the boundary of the second ε-neck and continue. This procedure can either terminate when we get into Ω_σ or an ε-cap, or go on infinitely, producing an ε-horn. The same procedure can be repeated for the other boundary component of the initial ε-neck. Therefore, we conclude that each ε-neck of $(\Omega, \overline{g}_{ij})$ is contained in a subset of Ω of one of the following types:

(4.3)
      (a) an ε-tube with boundary components in Ω_σ, or
      (b) an ε-cap with boundary in Ω_σ, or
      (c) an ε-horn with boundary in Ω_σ, or
      (d) a capped ε-horn, or
      (e) a double ε-horn.

Similarly, each ε-cap of $(\Omega, \overline{g}_{ij})$ is contained in a subset of Ω of either type (b) or type (d).

It is clear that there is a definite lower bound (depending on σ) for the volume of subsets of types (a), (b), (c), so there can be only a finite number of them. Thus we conclude that there is only a finite number of components of Ω, containing points of Ω_σ, and every such component has a finite number of ends, each being an ε-horn. By taking into account that Ω has no compact components, every component of Ω, containing no points of Ω_σ, is either a capped ε-horn, or a double ε-horn. Nevertheless, if we look at the solution for a slightly earlier time t, each ε-neck or ε-cap of $(M, g_{ij}(\cdot, t))$ is contained in a subset of types (a) and (b); while the ε-horns, capped ε-horns and double ε-horns, observed at the maximal time T, are connected together to form ε-tubes and ε-caps at the slightly earlier time t.

Hence, by looking at the solution for times just before T, we see that the topology of M^4 can be reconstructed as follows: take all components $\Omega_j, 1 \le j \le k$, of Ω which contain points of Ω_σ, truncate their ε-horns, and glue a finite collection of tubes $\mathbb{S}^3 \times \mathbb{I}$ and caps $\mathbb{B}^4$ or $\mathbb{RP}^4 \backslash \overline{\mathbb{B}^4}$ to the boundary components of truncated Ω_j. Thus M^4 is diffeomorphic

to a connected sum of $\overline{\Omega}_j$, $1 \leq j \leq k$, with a finite number of $\mathbb{S}^3 \times \mathbb{S}^1$ or
$\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$ (which correspond to gluing a tube to two boundary components
of the same Ω_j), and a finite number of $\mathbb{RP}^4$. Here $\overline{\Omega}_j$ denotes Ω_j with
each ε-horn one point compactified. (One might wonder why we do not
also cut other ε-tubes or ε-caps so that we can remove more volumes;
we will explain it a bit later.)

More geometrically, one can get $\overline{\Omega}_j$ in the following way: in every ε-
horn of Ω_j one can find an ε-neck, cut it along the middle three-sphere,
remove the horn-shaped end, and glue back a cap (i.e., a differentiable
four-ball). Thus to understand the topology of M^4, one need only to
understand the topologies of the compact four-manifolds $\overline{\Omega}_j, 1 \leq j \leq k$.

Recall that the four-manifold M^4 has no essential incompressible
space form; we now claim that each $\overline{\Omega}_j$ still has no essential incom-
pressible space form. Clearly, we only need to check the assertion that
if N is an essential incompressible space form in $\overline{\Omega}_j$, then N will be
also incompressible in M^4. After moving N slightly, we can choose N
such that $N \subset \Omega_j$. Then N can be regarded as a submanifold in M^4
(unaffected by the surgery). We now argue by contradiction. Suppose
$\gamma \subset N$ is a homotopically nontrivial curve which bounds a disk D in
M^4. We want to modify the map of disk D so that γ bound a new disk
in Ω_j, which will give the desired contradiction. Let $E_1, E_2, \ldots, E_m$ be
all the ε-horn ends of Ω_j, $S_1, S_2, \ldots, S_m \subset \Omega_j$ be the corresponding
cross spheres lying inside the ε-horn ends E_j respectively. Let us per-
turb the spheres $S_1, S_2, \ldots, S_m$ slightly so that they meet D transversely
in a finite number of simple closed curves (we only consider those S_j
with $S_j \cap D \neq \phi$). After removing those curves which are contained
in larger ones in D, we are left with a finite number of disjoint simple
closed curves, denoted by $C_1, C_2, \ldots, C_l$. We denote the enclosed disks
of $C_1, C_2, \ldots, C_l$ in D by $D_1, D_2, \ldots, D_l$. Since $\mathbb{S}^3$ is simply-connected,
each intersection curves in $S_1, S_2, \ldots, S_m$ can be shrunk to a point. So
by filling the holes $D_1, D_2, \ldots, D_l$, we obtain a new continuous map
from D to M^4 such that the image of $D_1 \cup D_2 \cdots \cup D_l$ is contained in
$S_1 \cup S_2 \cdots \cup S_m \subset \Omega_j$. On the other hand, since $D \backslash (D_1 \cup D_2 \cdots \cup D_l)$
is connected, γ(the image of ∂D) $\subset N$, we know that the image of
$D \backslash (D_1 \cup D_2 \cdots \cup D_l)$ must be contained in Ω_j. Therefore, γ bounds a
new disk in Ω_j. This proves that after the surgery, each $\overline{\Omega}_j$ still has no
essential incompressible space form.

As shown by Hamilton in Section D of [**21**], provided $\varepsilon > 0$ small
enough, one can perform the above surgery procedure carefully so that
the compact four-manifolds $\overline{\Omega}_j, 1 \leq j \leq k$, also have positive isotropic
curvature. Naturally, one can evolve each $\overline{\Omega}_j$ by the Ricci flow again
and carry out the same surgery procedure to produce a finite collection
of new compact four-manifolds with no essential incompressible space
form and with positive isotropic curvature. By repeating this procedure

indefinitely, it will be likely to give us the long time existence of a kind of "weak" solution to Ricci flow.

5. Ricci Flow with Surgery for Four-manifolds

We begin with an abstract definition of the solution to the Ricci flow with surgery which is adapted from [**32**].

Definition 5.1. Suppose we have a collection of compact four-dimensional smooth solutions $g_{ij}^{(k)}(t)$ to the Ricci flow on $M_k^4 \times [t_k^-, t_k^+)$ with no essential incompressible space form and with positive isotropic curvature, which go singular as $t \to t_k^+$ and where each manifold M_k^4 may be disconnected with only a finite number of connected components. Let $(\Omega_k, \overline{g}_{ij}^{(k)})$ be the limits of the corresponding solutions $g_{ij}^{(k)}(t)$ as $t \to t_k^+$. Suppose also that for each k we have $t_k^- = t_{k-1}^+$, and $(\Omega_{k-1}, \overline{g}_{ij}^{(k-1)})$ and $(M_k^4, g_{ij}^{(k)}(t_k^-))$ contain compact (possibly disconnected) four-dimensional submanifolds with smooth boundary which are isometric. Then by identifying these isometric submanifolds, we say it is a solution to the **Ricci flow with surgery** on the time interval which is the union of all $[t_k^-, t_k^+)$, and say the times t_k^+ are **surgery times**.

The procedure described in the last paragraph of the previous section gives us a solution to the Ricci flow with surgery. However, in order to understand the topology of the initial manifold from the solution to the Ricci flow with surgery, one encounters the following two difficulties:

(i) How to prevent the surgery times from accumulation?
(ii) How to get the long time behavior of the solution to the Ricci flow with surgery?

In view of this, it is natural to consider those solutions having "good" properties. Let ε be a fixed small positive number. We will only consider those solutions to the Ricci flow with surgery which satisfy the following a priori assumptions (with accuracy ε):

Pinching assumption: *There exist positive constants $\rho, \Lambda, P < +\infty$ such that there hold*

$$(5.1) \qquad a_1 + \rho > 0 \ and \ c_1 + \rho > 0,$$

$$(5.2) \quad \max\{a_3, b_3, c_3\} \le \Lambda(a_1 + \rho) \ and \ \max\{a_3, b_3, c_3\} \le \Lambda(c_1 + \rho),$$

and

$$(5.3) \qquad \frac{b_3}{\sqrt{(a_1 + \rho)(c_1 + \rho)}} \le 1 + \frac{\Lambda e^{Pt}}{\max\{\log \sqrt{(a_1 + \rho)(c_1 + \rho)}, 2\}},$$

everywhere.

Canonical neighborhood assumption (with accuracy ε): *For the given $\varepsilon > 0$, there exist two constants $C_1(\varepsilon)$, $C_2(\varepsilon)$ and a non-increasing positive function r on $[0, +\infty)$ such that for every point (x, t) where the scalar curvature $R(x, t)$ is at least $r^{-2}(t)$, there is an open neighborhood B, $B_t(x, \sigma) \subset B \subset B_t(x, 2\sigma)$ with $0 < \sigma < C_1(\varepsilon) R(x, t)^{-\frac{1}{2}}$, which falls into one of the following three categories:*

(a) *B is a **strong ε-neck** (in the sense that B is an ε-neck and it is the slice at time t of the parabolic neighborhood $\{(x', t') \mid x' \in B, t' \in [t - R(x, t)^{-1}, t]\}$, where the solution is well defined on the whole parabolic neighborhood and is, after scaling with factor $R(x, t)$ and shifting the time to zero, ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the corresponding subset of the evolving standard round cylinder $\mathbb{S}^3 \times \mathbb{R}$ with scalar curvature 1 at the time zero), or*

(b) *B is an ε-**cap**, or*

(c) *B is a compact four-manifold with positive curvature operator; furthermore, the scalar curvature in B at time t is between $C_2^{-1} R(x, t)$ and $C_2 R(x, t)$, and satisfies the gradient estimate*

(5.4)
$$|\nabla R| < \eta R^{\frac{3}{2}} \text{ and } \left| \frac{\partial R}{\partial t} \right| < \eta R^2,$$

and the volume of B in case (a) and case (b) satisfies

$$(C_2 R(x, t))^{-2} \leq \mathrm{Vol}_t(B).$$

Here C_1 and C_2 are some positive constants depending only on ε, and η is a universal positive constant.

Clearly, we may always assume the above C_1 and C_2 are twice bigger than the corresponding constants $C_1(\frac{\varepsilon}{2})$ and $C_2(\frac{\varepsilon}{2})$ in Theorem 3.8 with the accuracy $\frac{\varepsilon}{2}$.

The main purpose of this section is to construct a long-time solution to the Ricci flow with surgery which starts with an arbitrarily given compact four-manifold with no essential incompressible space form and with positive isotropic curvature, so that the a priori assumptions are satisfied and there are only a finite number of surgery times at each finite time interval. The construction will be given by an induction argument.

Firstly, for an arbitrarily given compact four-manifold $(M^4, g_{ij}(x))$ with no essential incompressible space form and with positive isotropic curvature, the Ricci flow with it as initial data has a maximal solution $g_{ij}(x, t)$ on $[0, T_0)$ with $T_0 < +\infty$. Without loss of generality, after a scaling on the initial metric, we may assume $T_0 > 1$. It follows from Lemma 2.1 and Theorem 4.1 that the a priori assumptions above hold for the smooth solution on $[0, T_0)$.

Suppose that we have a solution to the Ricci flow with surgery, with the given compact four-manifold $(M^4, g_{ij}(x))$ as initial datum, which is

defined on $[0, T)$ with $T < +\infty$, going singular at the time T, satisfies the a priori assumptions and has only a finite number of surgery times on $[0, T)$. Let Ω denote the set of all points where the curvature stays bounded as $t \to T$. As shown before, the gradient estimate (5.4) in the canonical neighborhood assumption implies that Ω is open and that $R(x, t) \to +\infty$ as $t \to T$ for x lying outside Ω. Moreover, as $t \to T$, the solution $g_{ij}(x, t)$ has a smooth limit $\overline{g}_{ij}(x)$ on Ω.

For $\delta > 0$ to be chosen much smaller than ε, we let $\sigma = \delta r(T)$ where $r(t)$ is the positive nonincreasing function in the definition of the canonical neighborhood assumption. We consider the corresponding compact set

$$\Omega_\sigma = \{x \in \Omega \mid \overline{R}(x) \leq \sigma^{-2}\}$$

where $\overline{R}(x)$ is the scalar curvature of $\overline{g}_{ij}$. If Ω_σ is empty, the manifold (near the maximal time T) is entirely covered by ε-tubes, ε-caps and compact components with positive curvature operator. Clearly, the number of compact components is finite. Then in this case the manifold (near the maximal time T) is diffeomorphic to the union of a finite number of $\mathbb{S}^4$, or $\mathbb{RP}^4$, or $\mathbb{S}^3 \times \mathbb{S}^1$, or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$, or a connected sum of them. Thus when Ω_σ is empty, the procedure stops here, and we say that the **solution becomes extinct**. We now assume Ω_σ is not empty. Every point $x \in \Omega \backslash \Omega_\sigma$ lies in one of the subsets of listing in (4.3), or in a compact component with positive curvature operator or in a compact component which is contained in $\Omega \backslash \Omega_\sigma$ and is diffeomorphic to $\mathbb{S}^4$, or $\mathbb{RP}^4$, or $\mathbb{S}^3 \times \mathbb{S}^1$, or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$. Note again that the number of compact components is finite. Let us throw away all the compact components lying $\Omega \backslash \Omega_\sigma$ or with positive curvature operator, and then consider all the components $\Omega_j, 1 \leq j \leq k$, of Ω which contain points of Ω_σ. (We will consider the components of $\Omega \backslash \Omega_\sigma$ consisting of capped ε-horns and double ε-horns later). We could perform Hamilton's surgical procedure in Section D of [**21**] at every horn of $\Omega_j, 1 \leq j \leq k$, so that the positive isotropic curvature condition and the pinching assumption is preserved.

Note that if we perform the surgeries at the necks with certain fixed accuracy ε on the high curvature region at each surgery time, then it is possible that the errors of surgeries may accumulate to a certain amount so that for some later time we can not recognize the structure of the very high curvature region. This prevents us from carrying out the process in finite time with finite steps. Hence in order to maintain the a priori assumptions *with the same accuracy* after surgery, we need to find sufficient "fine" necks in the ε-horns and to glue sufficient "fine" caps in the procedure of surgery. Note that $\delta > 0$ will be chosen much smaller than $\varepsilon > 0$. The following lemma gives us the "fine" necks in the ε-horns. (The corresponding result in three-dimension is Lemma 4.3 in [**32**].)

Now we explain why we only perform the surgeries in the horns with boundary in Ω_σ. At first sight, we should also cut off all those ε-tubes and ε-caps in the surgery procedure. But in general, we are not able to find a "finer" neck in an ε-tube or in ε-cap, and such surgeries at "rough" ε-necks will certainly loss some accuracy. This is the reason why we will only perform the surgeries in the ε-horns with boundary in Ω_σ.

Lemma 5.2. *Given $0 < \varepsilon < \frac{1}{100}, 0 < \delta < \varepsilon$ and $0 < T < +\infty$, there exists a radius $0 < h < \delta\sigma$, depending only on $\delta, r(T)$ and the pinching assumption, such that if we have a solution to the Ricci flow with surgery, with a compact four-manifold $(M^4, g_{ij}(x))$ with no essential incompressible space form and with positive isotropic curvature as initial data, defined on $[0, T)$, going singular at the time T, and satisfying the a priori assumptions and having only a finite number of surgery times on $[0, T)$, then for each point x with $h(x) = \overline{R}^{-\frac{1}{2}}(x) \le h$ in an ε-horn of $(\Omega, \overline{g}_{ij})$ with boundary in Ω_σ, the neighborhood $B_T(x, \delta^{-1}h(x)) = \{y \in \Omega | dist_{\overline{g}_{ij}}(y, x) \le \delta^{-1}h(x)\}$ is a strong δ-neck (i.e., $\{(y, t) \mid y \in B_T(x, \delta^{-1}h(x)), t \in [T - h^2(x), T]\}$ is, after scaling with factor $h^{-2}(x)$, δ-close (in $C^{[\delta^{-1}]}$ topology) to the corresponding subset of the evolving standard round cylinder $\mathbb{S}^3 \times \mathbb{R}$ over the time interval $[-1, 0]$ with scalar curvature 1 at time zero).*

Proof. We argue as in [**32**] by contradiction. Suppose that there exists a sequence of solutions $g_{ij}^{(k)}(\cdot, t), k = 1, 2, \ldots$, to the Ricci flow with surgery, satisfying the a priori assumptions, defined on $[0, T)$ with limits $(\Omega^k, \overline{g}_{ij}^{(k)}), k = 1, 2, \ldots$, as $t \to T$, and there exist points x_k, lying inside an ε-horn of Ω^k, which contains the points of Ω_σ^k, and having $h(x_k) \to 0$ as $k \to +\infty$ such that the neighborhood $B_T(x_k, \delta^{-1}h(x_k))$ are not strong δ-necks.

Let $\widetilde{g}_{ij}^{(k)}(\cdot, t)$ be the rescaled solutions by the factor $\overline{R}(x_k) = h^{-2}(x_k)$ around (x_k, T). We will show that a sequence of $\widetilde{g}_{ij}^{(k)}(\cdot, t)$ converges to the evolving round $\mathbb{R} \times \mathbb{S}^3$, which gives the desired contradiction.

Note that $\widetilde{g}_{ij}^{(k)}(\cdot, t), k = 1, 2, \ldots$, are modified by surgery. We can not apply Hamilton's compactness theorem directly since it states only for smooth solutions. For each (unrescaled) surgical solution $\widetilde{g}_{ij}^{(k)}(\cdot, t)$, we pick a point z_k, with $\bar{R}(z_k) = 2C_2^2(\varepsilon)\sigma^{-2}$, in the ε-horn of $(\Omega^k, \bar{g}_{ij}^{(k)})$ with boundary in Ω_σ^k, where $C_2(\varepsilon)$ is the positive constant in the canonical neighborhood assumption. From the definition of ε-horn and the canonical neighborhood assumption, we know that each point x lying inside the ε-horn of $(\Omega^k, \bar{g}_{ij}^{(k)})$ with $d_{\bar{g}_{ij}^{(k)}}(x, \Omega_\sigma^k) \ge d_{\bar{g}_{ij}^{(k)}}(z_k, \Omega_\sigma^k)$ has a strong ε-neck as its canonical neighborhood. Since $h(x_k) \to 0$, each

x_k lies deeply inside the ε-horn. Thus for each positive $A < +\infty$, the rescaled (surgical) solutions $\widetilde{g}_{ij}^{(k)}(\cdot, t)$ with the marked origins x_k over the geodesic balls $B_{\widetilde{g}_{ij}^{(k)}(\cdot,0)}(x_k, A)$, centered at x_k of radii A (with respect to the metrics $\widetilde{g}_{ij}^{(k)}(\cdot, 0)$), will be smooth on some uniform (size) small time intervals for all sufficiently large k, if the curvatures of the rescaled solutions $\widetilde{g}_{ij}^{(k)}$ at $t = 0$ in $B_{\widetilde{g}_{ij}^{(k)}(\cdot,0)}(x_k, A)$ are uniformly bounded. In such situation, the Hamilton's compactness theorem is applicable. We can now apply the same argument as in Step 2 of the proof of Theorem 4.1 to conclude that the curvatures of the rescaled solutions $\widetilde{g}_{ij}^{(k)}(\cdot, t)$ at the time T stay uniformly bounded at bounded distances from x_k; otherwise we get a piece of a non-flat nonnegative curved metric cone as a blow-up limit, which would contradict the Hamilton strong maximum principle [**16**]. Hence as before we can get a C_{loc}^∞ limit $\widetilde{g}_{ij}^{(\infty)}(\cdot, t)$, defined on a space-time set which is relatively open in the half space-time $\{t \le T\}$ and contains the time slice $\{t = T\}$, from the rescaled solutions $\widetilde{g}_{ij}^{(k)}(\cdot, t)$.

By the pinching assumption, the limit is a complete manifold with the restricted pinching condition (2.4) and with nonnegative curvature operator. Since x_k was contained in an ε-horn with boundary in Ω_σ^k, and $h(x_k)/\sigma \to 0$, the limiting manifold has two ends. Thus by Toponogov splitting theorem, it admits a (maybe not round at this moment) metric splitting $\mathbb{R} \times \mathbb{S}^3$ because x_k was the center of a strong ε-neck. We further apply the restricted isotropic curvature pinching condition (2.4) and contracted second Bianchi identity as before to conclude that the factor $\mathbb{S}^3$ must be round at time 0. By combining with the canonical neighborhood assumption, we see that the limit is defined on the time interval $[-1, 0]$. By Toponogov splitting theorem, the splitting $\mathbb{R} \times \mathbb{S}^3$ is at each time $t \in [-1, 0]$; so the limiting solution is just the standard evolving round cylinder. This is a contradiction. We finish the proof of Lemma 5.2.   q.e.d.

The property in the above lemma that the radius h depends only on δ, the time T and the pinching assumption, independent of the surgical solution, is crucial; otherwise we will not be able to cut off enough volume at each surgery to guarantee the number of surgeries being finite in each finite time interval.

Remark. The proof of Lemma 5.2 actually proves a more stronger result, which will be used in the proof of Proposition 5.4.

For any $\delta > 0$, there exists a radius $0 < h < \delta\sigma$, depending only on $\delta, r(T)$ and the pinching assumption, such that for each point x with $h(x) = \overline{R}^{-\frac{1}{2}}(x) \le h$ in an ε-horn of $(\Omega, \overline{g}_{ij})$ with boundary in Ω_σ,

$\{(y,t) \mid y \in B_T(x, \delta^{-1}h(x)), t \in [T - \delta^{-2}h^2(x), T]\}$ *is, after scaling with* *factor* $h^{-2}(x)$, δ-*close (in* $C^{[\delta^{-1}]}$ *topology) to the corresponding subset* *of the evolving standard round cylinder* $\mathbb{S}^3 \times \mathbb{R}$ *over the time interval* $[-\delta^{-2}, 0]$ *with scalar curvature* 1 *at the time zero.*

The reason is as follows. Let us use the notation in the proof the Lemma 5.2 and argue by contradiction. Note that the scalar curvature of the limit at time $t = -1$ is $\frac{1}{1-\frac{2}{3}(-1)}$. Since $h(x^k)/\rho \to 0$, each point in the limiting manifold at time $t = -1$ also has a strong ε-neck as its canonical neighborhood. Thus the limit is defined at least on the time interval $[-2, 0]$. Inductively, suppose the limit is defined on the time interval $[-m, 0]$ with bounded curvature for some positive integer $-m$; then by the isotropic pinching condition, Toponogov splitting theorem and the evolution equation of the scalar curvature on the round $\mathbb{R} \times \mathbb{S}^3$, we see that $R = \frac{1}{1+\frac{2}{3}m}$ at time $-m$. Since $h(x^k)/\rho \to 0$, each point in the limiting manifold at time $t = -m$ has also a strong ε-neck as its canonical neighborhood, we see that the limit is defined at least on the time interval $[-(m + 1), 0]$ with bounded curvature. So by induction we prove that the limit exists on the ancient time interval $(-\infty, 0]$. Therefore the limit is the evolving round cylinder $\mathbb{S}^3 \times \mathbb{R}$ over the time interval $(-\infty, 0]$, which gives the desired contradiction.

To specialize our surgery, we now fix a standard capped infinite cylinder for $n = 4$ as follows. Consider the semi-infinite standard round cylinder $N_0 = \mathbb{S}^3 \times (-\infty, 4)$ with the metric g_0 of scalar curvature 1. Denote by z the coordinate of the second factor $(-\infty, 4)$. Let f be a smooth nondecreasing convex function on $(-\infty, 4)$ defined by

$$\begin{cases} f(z) = 0, & z \leq 0, \\[2mm] f(z) = ce^{-\frac{D}{z}}, & z \in (0, 3], \\[2mm] f(z) \text{ is strictly convex on } z \in [3, 3.9], \\[2mm] f(z) = -\frac{1}{2}\log(16 - z^2), & z \in [3.9, 4), \end{cases}$$

where the small (positive) constant c and big (positive) constant D will be determined later. Let us replace the standard metric g_0 on the portion $\mathbb{S}^3 \times [0, 4)$ of the semi-infinite cylinder by $\hat{g} = e^{-2f}g_0$. Then the resulting metric $\hat{g}$ will be smooth on $\mathbb{R}^4$ obtained by adding a point to $\mathbb{S}^3 \times (-\infty, 4)$ at $z = 4$. We denote by $C(c, D) = (\mathbb{R}^4, \hat{g})$. Clearly, $C(c, D)$ is a standard capped infinite cylinder.

We next use a compact portion of the standard capped infinite cylinder $C(c, D)$ and the δ-neck obtained in Lemma 5.2 to perform the following surgery due to Hamilton [21].

Consider the solution metric $\bar{g}$ at the maximal time $T < +\infty$. Take an ε-horn with boundary in Ω_ρ. By Lemma 5.2, there exists a δ-neck N

of radius $0 < h < \delta\rho$ in the ε-horn. By definition, $(N, h^{-2}\bar{g})$ is δ-close (in $C^{[\delta^{-1}]}$ topology) to the standard round neck $\mathbb{S}^3 \times \mathbb{I}$ of scalar curvature 1 with $\mathbb{I} = (-\delta^{-1}, \delta^{-1})$. The parameter $z \in \mathbb{I}$ induces a function on the δ-neck N.

Let us cut the δ-neck N along the middle (topological) three-sphere $N \bigcap \{z = 0\}$. Without loss of generality, we may assume that the right hand half portion $N \bigcap \{z \geq 0\}$ is contained in the horn-shaped end. Let φ be a smooth bump function with $\varphi = 1$ for $z \leq 2$, and $\varphi = 0$ for $z \geq 3$. Construct a new metric $\tilde{g}$ on a (topological) four-ball $\mathbb{B}^4$ as follows:

$$
\tilde{g} = \begin{cases}
\bar{g}, & z = 0, \\[2mm]
e^{-2f}\bar{g}, & z \in [0,2], \\[2mm]
\varphi e^{-2f}\bar{g} + (1 - \varphi)e^{-2f}h^2 g_0, & z \in [2,3], \\[2mm]
h^2 e^{-2f} g_0, & z \in [3,4].
\end{cases}
$$

The surgery is to replace the horn-shaped end by the cap $(\mathbb{B}^4, \tilde{g})$. The following lemma, due to Hamilton [21], determines the constants c and D in the δ-cutoff surgery so that the pinching assumption is preserved under the surgery.

Lemma 5.3 (Hamilton [21] D3.1. Justification of the pinching assumption)**.** *There are universal positive constants δ_0, c_0 and D_0 such that for any $\tilde{T}$ there is a constant $h_0 > 0$ depending on the initial metric and $\tilde{T}$ such that if we take a δ-cutoff surgery at a δ-neck of radius h at time $T \leq \tilde{T}$ with $\delta < \delta_0$ and $h^{-2} \geq h_0^{-2}$, then we can choose $c = c_0$ and $D = D_0$ in the definition of $f(z)$ such that after the surgery, there still holds the pinching condition (2.1) (2.2) (2.3) :*

$$a_1 + \rho > 0 \ and \ c_1 + \rho > 0,$$

$$\max\{a_3, b_3, c_3\} \leq \Lambda(a_1 + \rho) \ and \ \max\{a_3, b_3, c_3\} \leq \Lambda(c_1 + \rho),$$

and

$$\frac{b_3}{\sqrt{(a_1 + \rho)(c_1 + \rho)}} \leq 1 + \frac{\Lambda e^{Pt}}{\max\{\log\sqrt{(a_1 + \rho)(c_1 + \rho)}, 2\}}$$

at all points at time T. Moreover, after the surgery, any metric ball of radius $\delta^{-\frac{1}{2}}h$ with center near the tip (i.e., the origin of the attached cap) is, after scaling with factor h^{-2}, $\delta^{\frac{1}{2}}$-close the corresponding ball of the standard capped infinite cylinder $C(c_0, D_0)$.

We call the above procedure as a δ-**cutoff surgery**. Since there are only a finite number of horns with their other ends connected to Ω_σ, we only need to *perform a finite number of such δ-cutoff surgeries at the time T*. Besides those horns, there could be capped horns, double

horns and compact components lying in $\Omega \setminus \Omega_\sigma$ or with positive curvature operator. As explained before, capped horns and double horns are connected with horns to form tubes or capped tubes at any time slightly before T. Thus when we truncated the horns at the δ-cutoff surgeries, we actually had removed these together with the horn-shaped ends away. So *we can regard the capped horns and double horns (of $\Omega \setminus \Omega_\sigma$) as extinct and throw them away at the time T*. Remember that *we have thrown away all the compact components lying in $\Omega \setminus \Omega_\sigma$ or with positive curvature operator*. Each such compact component is diffeomorphic to $\mathbb{S}^4$, or $\mathbb{RP}^4$, or $\mathbb{S}^3 \times \mathbb{S}^1$, or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$, and the number of compact components is finite. Thus we actually throw a finite number of $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$ at the time T also. (Note that we allow that the manifold may be disconnected before and after the surgeries.) Let us agree to **declare extinct every compact component with positive curvature operator or lying in $\Omega \setminus \Omega_\sigma$**; in particular, that allows to exclude the components with positive curvature operator from the list of canonical neighborhoods.

Summarily, our surgery at the time T consists of the following four procedures:

(1) *perform δ-cutoff surgeries for all ε-horns which have the other ends connected to Ω_σ,*

(2) *declare extinct every compact component which has positive curvature operator,*

(3) *throw away all capped horns and double horns lying in $\Omega \setminus \Omega_\sigma$,*

(4) *declare extinct every compact component lying in $\Omega \setminus \Omega_\sigma$.*

After the surgery at the time T, the pinching assumption still holds for the surgically modified manifolds. With this (maybe disconnected) surgically modified manifold as initial data, we now continue our solution until it becomes singular for the next time $T'(> T)$. Therefore we have extended the solution to the Ricci flow with surgery, originally defined on $[0, T)$, to the new time interval $[0, T')$ (with $T' > T$). Moreover, as long as $0 < \delta \leq \delta_0$, the solution with δ-cutoff surgeries on the new time interval $[0, T')$ still has positive isotropic curvature and no essential incompressible space form, and from [**21**] and Lemma 5.3 it still satisfies the pinching assumption.

Denote the minimum of the scalar curvature at time t by $R_{\min}(t) > 0$. Since the δ-cutoff surgeries occur at the points lying deeply in the ε-horns, the minimum of the scalar curvature $R_{\min}(t)$ of the solution at each time-slice is achieved in the region unaffected by the surgeries. Thus we know from the evolution equation of the scalar curvature that

$$\frac{d}{dt}R_{\min}(t) \geq \frac{1}{2}R_{\min}^2(t).$$

By integrating this inequality, we conclude that the maximal time T of any solution to the Ricci flow with δ-cutoff surgeries must be bounded by $2/R_{\min}(0) < +\infty$. Let $\tilde{T} = 2/R_{\min}(0)$ in Lemma 5.3, then there is a constant h_0 determined by $\tilde{T}$. Set $\bar{\delta} = \frac{1}{2}R_{\min}(0)^{\frac{1}{2}}h_0$. We know that if we perform the δ-cutoff surgery with $\delta < \min\{\bar{\delta}, \delta_0\}$, then the pinching assumptions (5.1), (5.2), (5.3) are satisfied for the solution to the Ricci flow with δ-cutoff surgery. Next we make further restrictions on δ to justify the canonical neighborhood assumption. Clearly, we only need to check the following assertion, which extends the crucial Proposition 5.1 of Perelman [32] to four-dimension.

Proposition 5.4 (Justification of the canonical neighborhood assumption). *Given a compact four-manifold with positive isotropic curvature and no essential incompressible space form and given $\varepsilon > 0$, there exist decreasing sequences $\varepsilon > \tilde{r}_j > 0$, $\kappa_j > 0$, $\min\{\varepsilon^2, \delta_0, \bar{\delta}\} > \tilde{\delta}_j > 0$, $j = 1, 2, \ldots$, with the following property. Define a positive function $\tilde{\delta}(t)$ on $[0, +\infty)$ by $\tilde{\delta}(t) = \tilde{\delta}_j$ when $t \in [(j-1)\varepsilon^2, j\varepsilon^2)$. Suppose we have a solution to the Ricci flow with surgery, with the given four-manifold as initial datum defined on the time interval $[0, T)$ and with a finite number of δ-cutoff surgeries such that any δ-cutoff surgery at a time $t \in (0, T)$ with $\delta = \delta(t)$ satisfies $0 < \delta(t) \leq \tilde{\delta}(t)$. Then on each time interval $[(j-1)\varepsilon^2, j\varepsilon^2] \bigcap [0, T)$, the solution satisfies the κ_j-noncollapsing condition on all scales less than ε and the canonical neighborhood assumption (with accuracy ε) with $r = \tilde{r}_j$.*

Here and in the following, we call a (four-dimensional) solution $g_{ij}(t)$, $0 \leq t < T$, to the Ricci flow with surgery is κ-**noncollapsed** at a point (x_0, t_0) on the scales less than ρ (for some $\kappa > 0, \rho > 0$) if it satisfies the following property: whenever $r < \rho$ and

$$|Rm(x, t)| \leq r^{-2}$$

for all those $(x, t) \in P(x_0, t_0, r, -r^2) = \{(x', t') \mid x' \in B_{t'}(x_0, r), t' \in [t_0 - r^2, t_0]\}$, for which the solution is defined, we have

$$\mathrm{Vol}_{t_0}(B_{t_0}(x_0, r)) \geq \kappa r^4.$$

Before we give the proof of the proposition, we need to check the κ-noncollapsing condition which extends the Lemma 5.2 of Perelman [32] to four-dimension.

Lemma 5.5. *For a given compact four-manifold with positive isotropic curvature and no essential incompressible space form, and given $\varepsilon > 0$, suppose we have constructed the sequences, satisfying the above proposition for $1 \leq j \leq \ell$. Then there exists $\kappa > 0$, such that for any r, $0 < r < \varepsilon$, one can find $\tilde{\delta}$ with $0 < \tilde{\delta} < \min\{\varepsilon^2, \delta_0, \bar{\delta}\}$, which depends on r, ε and may also depend on the already constructed sequences, with*

the following property. Suppose we have a solution, with the given four-manifold as initial data, to the Ricci flow with surgery defined on a time interval $[0, T]$ with $\ell \varepsilon^2 \leq T < (\ell + 1)\varepsilon^2$ such that the assumptions and conclusions of Proposition 5.4 hold on $[0, \ell \varepsilon^2)$, the canonical neighborhood assumption (with accuracy ε) with r holds on $[\ell \varepsilon^2, T]$, and each $\delta(t)$-cutoff surgery in the time interval $t \in [(\ell-1)\varepsilon^2, T]$ has $0 < \delta(t) < \tilde{\delta}$. Then the solution is κ-noncollapsed on $[0, T]$ for all scales less than ε.

Proof. Consider a parabolic neighborhood $P(x_0, t_0, r_0, -r_0^2) = \{(x, t) | x \in B_t(x_0, r_0), t \in [t_0 - r_0^2, t_0]\}$, with $\ell \varepsilon^2 \leq t_0 \leq T$, and $0 < r_0 \leq \varepsilon$, where the solution satisfies $|Rm| \leq r_0^{-2}$ whenever it is defined. We will prove that $\mathrm{Vol}_{t_0}(B_{t_0}(x_0, r_0)) \geq \kappa r_0^4$.

Let η be the universal positive constant in the definition of the canonical neighborhood assumption. Without loss of generality, we always assume $\eta \geq 10$. Firstly, we want to show that one may assume $r_0 \geq \frac{1}{2\eta} r$.

Obviously, the curvature satisfies the estimate

$$|Rm(x, t)| \leq 20 r_0^{-2},$$

for those $(x, t) \in P(x_0, t_0, \frac{1}{2\eta} r_0, -\frac{1}{8\eta} r_0^2) = \{(x, t) \mid x \in B_t(x_0, \frac{1}{2\eta} r_0), t \in [t_0 - \frac{1}{8\eta} r_0^2, t_0]\}$ for which the solution is defined. When $r_0 < \frac{1}{2\eta} r$, we can enlarge r_0 to some $r_0' \in [r_0, r]$ so that

$$|Rm| \leq 20 r_0'^{-2}$$

on $P(x_0, t_0, \frac{1}{2\eta} r_0', -\frac{1}{8\eta} r_0'^2)$ (whenever it is defined), and either the equality holds somewhere or $r_0' = r$.

In the case that the equality holds somewhere, it follows from the pinching assumption that we have

$$R > 10 r_0'^{-2}$$

somewhere in $P(x_0, t_0, \frac{1}{2\eta} r_0', -\frac{1}{8\eta} r_0'^2)$. Here, without loss of generality, we have assumed r is suitably small. Then by the gradient estimates in the definition of the canonical neighborhood assumption, we know

$$R(x_0, t_0) > r_0'^{-2} \geq r^{-2}.$$

Hence the desired noncollapsing estimate in this case follows directly from the canonical neighborhood assumption. (Recall that we have excluded every component which has positive sectional curvature in the surgery procedure and then excluded them from the list of canonical neighborhoods. Here we also used the standard volume comparison when the canonical neighborhood is an ε-cap.)

While in the case that $r_0' = r$, we have the curvature bound

$$|Rm(x, t)| \leq \left(\frac{1}{2\eta} r\right)^{-2},$$

for those $(x, t) \in P(x_0, t_0, \frac{1}{2\eta}r, -(\frac{1}{2\eta}r)^2) = \{(x, t) \mid x \in B_t(x_0, \frac{1}{2\eta}r), t \in [t_0 - (\frac{1}{2\eta}r)^2, t_0]\}$ for which the solution is defined. It follows from the standard volume comparison that we only need to verify the noncollapsing estimate for $r_0 = \frac{1}{2\eta}r$. Thus we have reduced the proof to the case $r_0 \geq \frac{1}{2\eta}r$.

The reduced distance from (x_0, t_0) is

$$l(q, \tau) = \frac{1}{2\sqrt{\tau}} \inf \left\{ \int_0^\tau \sqrt{s}(R(\gamma(s), t_0 - s) + |\dot{\gamma}(s)|^2_{g_{ij}(t_0-s)})ds \mid \right.$$
$$\left. \gamma(0) = x_0, \gamma(\tau) = q \right\}$$

where $\tau = t_0 - t$ with $t < t_0$. Firstly, we need to check that the minimum of the reduced distance is achieved by curves unaffected by surgery. According to Perelman [32], we call a space-time curve in the solution track **admissible** if it stays in the space-time region unaffected by surgery, and we call a space-time curve in the solution track a **barely admissible curve** if it is on the boundary of the set of admissible curves. The following assertion gives a big lower bound for the reduced lengths of barely admissible curves.

Claim 1. For any $L < +\infty$ one can find $\widetilde{\delta} = \widetilde{\delta}(L, r, \widetilde{r}_\ell, \varepsilon) > 0$ with the following property. Suppose that we have a curve γ, parametrized by $t \in [T_0, t_0], (\ell - 1)\varepsilon^2 \leq T_0 < t_0$, such that $\gamma(t_0) = x_0$, T_0 is a surgery time and $\gamma(T_0)$ lies in a $4h$-collar of the middle three-sphere of a δ-neck with the radius h obtained in Lemma 5.2, where the δ-cutoff surgery was taken. Suppose also that each $\delta(t)$-cutoff surgery in the time interval $t \in [(\ell - 1)\varepsilon^2, T]$ has $0 < \delta(t) < \widetilde{\delta}$. Then we have an estimate

$$(5.5) \quad \int_0^{t_0-T_0} \sqrt{\tau}(R(\gamma(t_0 - \tau), t_0 - \tau) + |\dot{\gamma}(t_0 - \tau)|^2_{g_{ij}(t_0-\tau)})d\tau \geq L,$$

where $\tau = t_0 - t \in [0, t_0 - T_0]$.

Before we can verify this assertion, we need to do some premilary work.

Let O be the point near $\gamma(T_0)$ which corresponds to the center of the (rotationally symmetric) capped infinite round cylinder. Recall from Lemma 5.3 that a metric ball of radius $\delta^{-\frac{1}{2}}h$ at time T_0 centered at O is, after scaling with factor h^{-2}, $\delta^{\frac{1}{2}}$-close (in $C^{[\delta^{-\frac{1}{2}}]}$ topology) to the corresponding ball in the capped infinite round cylinder. We need to consider the solutions to the Ricci flow with the capped infinite round cylinder (with scalar curvature 1 outsider some compact set) as initial data and we require that the solutions also have bounded curvature; we call such a solution a **standard solution** as in [32]. From Shi [36], we know such a solution exists. The uniqueness of the Ricci flow for

compact manifolds is well-known (see for example, Section 6 of [**20**]). In [**11**], we prove a uniqueness theorem which states that if the initial data is a complete noncompact Riemannian manifold with bounded curvature, then the solution to the Ricci flow in the class of complete solutions with bounded curvature is unique. Thus the standard solution with a capped infinite round cylinder as initial data is unique. In the appendix, we will show that the standard solution exists on the time interval $[0, \frac{3}{2})$ and has nonnegative curvature operator, and its scalar curvature satisfies

$$(5.6) \qquad\qquad R(x, t) \geq \frac{C^{-1}}{\frac{3}{2} - t},$$

everywhere for some positive constant C.

For any $0 < \theta < \frac{3}{2}$, let Q be the maximum of the scalar curvature of the standard solution in the time interval $[0, \theta]$ and let $\triangle t = (T_1 - T_0)/N < \varepsilon \eta^{-1} Q^{-1} h^2$ with $T_1 = \min\{t_0, T_0 + \theta h^2\}$ and η given in the canonical neighborhood assumption. Set $t_k = T_0 + k \triangle t, k = 1, \ldots, N$.

Note that the ball $B_{T_0}(O, A_0 h)$ at time T_0 with $A_0 = \delta^{-\frac{1}{2}}$ is, after scaling with factor h^{-2}, $\delta^{\frac{1}{2}}$-close to the corresponding ball in the capped infinite round cylinder. Assume first that for each point in $B_{T_0}(O, A_0 h)$, the solution is defined on $[T_0, t_1]$. By the gradient estimate (5.4) in the canonical neighborhood assumption and the choice of $\triangle t$ we have a uniform curvature bound on this set for h^{-2}-scaled metric. Then by the uniqueness theorem in [**11**], if $\delta^{\frac{1}{2}} \to 0$ (i.e., $A_0 = \delta^{-\frac{1}{2}} \to +\infty$), the solution with h^{-2}-scaled metric will converge to the standard solution in C_{loc}^{∞} topology. Therefore we can find A_1, depending only on A_0 and tending to infinity with A_0, such that the solution in the parabolic region $P(O, T_0, A_1 h, t_1 - T_0) = \{(x, t)| x \in B_t(O, A_1 h), t \in [T_0, T_0 + (t_1 - T_0)]\}$ is, after scaling with factor h^{-2} and shifting time T_0 to zero, A_1^{-1}-close to the corresponding subset in the standard solution. In particular, the scalar curvature on this subset does not exceed $2Qh^{-2}$. Now if for each point in $B_{T_0}(O, A_1 h)$ the solution is defined on $[T_0, t_2]$, then we can repeat the procedure, defining A_2, such that the solution in the parabolic region $P(O, T_0, A_2 h, t_2 - T_0) = \{(x, t)| x \in B_t(p, A_2 h), t \in [T_0, T_0 + (t_2 - T_0)]\}$ is, after scaling with factor h^{-2} and shifting time T_0 to zero, A_2^{-1}-close to the corresponding subset in the standard solution. Again, the scalar curvature on this subset still does not exceed $2Qh^{-2}$. Continuing this way, we eventually define A_N. Note that N depends only on θ. Thus for arbitrarily given $A > 0$ (to be determined), we can choose $\widetilde{\delta}(A, \theta, \varepsilon) > 0$ such that as $\delta < \widetilde{\delta}(A, \theta, \varepsilon)$, and assuming that for each point in $B_{T_0}(O, A_{(N-1)} h)$ the solution is defined on $[T_0, T_1]$, we have $A_0 > A_1 > \cdots > A_N > A$, and the solution in $P(O, T_0, Ah, T_1 - T_0) = \{(x, t)| x \in B_t(O, Ah), t \in [T_0, T_1]\}$ is, after scaling with factor h^{-2} and

shifting time T_0 to zero, A^{-1}-close to the corresponding subset in the standard solution.

Now assume that there exist some k $(1 \leq k \leq N - 1)$ and a surgery time $t^+ \in (t_k, t_{k+1}]$(or $t^+ \in (T_0, t_1]$) such that on $B_{T_0}(O, A_k h)$ the solution is defined on $[T_0, t^+)$, but for some point of this ball it is not defined past t^+. Clearly the above argument also shows that the parabolic region $P(O, T_0, A_{k+1}h, t^+ - T_0) = \{(x,t)|x \in B_t(x, A_{k+1}h), t \in [T_0, t^+)\}$ is, after scaling with factor h^{-2} and shifting time T_0 to zero, A_{k+1}^{-1}-close to the corresponding subset in the standard solution. In particular, as the time tends to t^+, the ball $B_{T_0}(O, A_{k+1}h)$ keeps on looking like a cap. Since the scalar curvature on the set $B_{T_0}(O, A_k h) \times [T_0, t_k]$ does not exceed $2Qh^{-2}$, it follows from the pinching assumption, the gradient estimates in the canonical neighborhood assumption and the evolution equation of the metric that the diameter of the set $B_{T_0}(O, A_k h)$ at any time $t \in [T_0, t^+)$ is bounded from above by $4\delta^{-\frac{1}{2}}h$. These imply that no point of the ball $B_{T_0}(O, A_k h)$ at any time near t^+ can be the center of a δ-neck for any $0 < \delta < \widetilde{\delta}(A, \theta, \varepsilon)$ with $\widetilde{\delta}(A, \theta, \varepsilon) > 0$ small enough, since $4\delta^{-\frac{1}{2}}h << \delta^{-1}h$. However, the solution disappears somewhere in the ball $B_{T_0}(O, A_k h)$ at the time t^+ because of a δ-cutoff surgery and the surgery is always done along the middle three-sphere of a δ-neck. So the set $B_{T_0}(O, A_k h)$ at the time t^+ is a part of a capped horn. (Recall that we have declared extinct every compact component with positive curvature operator or lying in $\Omega \setminus \Omega_\sigma$). And then for each point of $B_{T_0}(O, A_k h)$ the solution terminates at t^+.

The above observations will give us the following consequence.

Claim 2. For any $\widetilde{L} < +\infty$, one can find $A = A(\widetilde{L}) < +\infty$ and $\theta = \theta(\widetilde{L}), 0 < \theta < \frac{3}{2}$, with the following property. Suppose γ is a smooth curve in the set $B_{T_0}(O, Ah)$, parametrized by $t \in [T_0, T_\gamma]$, such that $\gamma(T_0) \in B_{T_0}(O, \frac{1}{2}Ah)$ and either $T_\gamma = T_1$ and the solution on $B_{T_0}(O, Ah)$ exists up to the time interval $[T_0, T_1]$ with $T_1 = \min\{t_0, T_0 + \theta h^2\} < t_0$, or $T_\gamma < T_1$ and $\gamma(T_\gamma) \in \partial B_{T_0}(O, Ah)$. Then as $\delta < \widetilde{\delta}(A, \theta, \varepsilon)$ chosen before, there holds

(5.7)
$$\int_{T_0}^{T_\gamma} (R(\gamma(t), t) + |\dot{\gamma}(t)|^2_{g_{ij}(t)})dt > \widetilde{L}.$$

Indeed, we know from the estimate (5.6) that on the standard solution,

$$\int_0^\theta Rdt \geq \text{const.} \int_0^\theta \left(\frac{3}{2} - t\right)^{-1} dt$$
$$= -\text{const.} \log\left(1 - \frac{2\theta}{3}\right).$$

By choosing $\theta = \theta(\widetilde{L})$ sufficiently close to $\frac{3}{2}$, we have the desired estimate on the standard solution.

If $T_\gamma = T_1 < t_0$ and the solution on $B_{T_0}(O, Ah)$ exists up to the time interval $[T_0, T_1]$, the solution in the parabolic region $P(O, T_0, Ah, T_1 - T_0) = \{(x, t) | x \in B_t(O, Ah), t \in [T_0, T_1]\}$ is, after scaling with factor h^{-2} and shifting time T_0 to zero, A^{-1}-close to the corresponding subset in the standard solution. Then we have

$$\int_{T_0}^{T_\gamma} (R(\gamma(t), t) + |\dot{\gamma}(t)|^2_{g_{ij}(t)}) dt \geq \text{const.} \int_0^\theta \left(\frac{3}{2} - t\right)^{-1} dt$$

$$= -\text{const.} \log\left(1 - \frac{2\theta}{3}\right),$$

which gives the desired estimate in this case.

Meanwhile if $T_\gamma < T_1$ and $\gamma(T_\gamma) \in \partial B_{T_0}(O, Ah)$, we see that the solution on $B_{T_0}(O, A_0 h)$ exists up to the time interval $[T_0, T_\gamma]$ and is, after scaling, A^{-1}-close to corresponding set in the standard solution. Let $\theta = \theta(\widetilde{L})$ be chosen as above and set $Q = Q(\widetilde{L})$ to be the maximum of the scalar curvature of the standard solution in the time interval $[0, \theta]$. On the standard solution, we can choose $A = A(\widetilde{L})$ so large that for each $t \in [0, \theta]$,

$$\begin{aligned} d_t(O, \partial B_0(O, A)) &\geq d_0(O, \partial B_0(O, A)) - 4(Q+1)t \\ &\geq A - 4(Q+1)\theta \\ &\geq \frac{4}{5} A, \end{aligned}$$

and

$$d_t\left(O, \partial B_0\left(O, \frac{A}{2}\right)\right) \leq \frac{A}{2},$$

where we used Lemma 8.3 of [**31**] in the first inequality. Now our solution in the subset $B_{T_0}(O, Ah)$ up to the time interval $[T_0, T_\gamma]$ is (after scaling) A^{-1}-close to the corresponding subset in the standard solution. This implies

$$\frac{1}{5} Ah \leq \int_{T_0}^{T_\gamma} |\dot{\gamma}(t)|_{g_{ij}(t)} \leq \left(\int_{T_0}^{T_\gamma} |\dot{\gamma}(t)|^2_{g_{ij}(t)} dt\right)^{\frac{1}{2}} (T_\gamma - T_0)^{\frac{1}{2}}$$

and then

$$\int_{T_0}^{T_\gamma} \left(R(\gamma(t), t) + |\dot{\gamma}(t)|^2_{g_{ij}(t)}\right) dt \geq \frac{A^2}{25\theta} > \widetilde{L},$$

by choosing $A = A(\widetilde{L})$ large enough. This proves Claim 2.

We now use the above Claim 2 to verify Claim 1. Since $r_0 \geq \frac{1}{2\eta} r$ and $|R_m| \leq r_0^{-2}$ on $P(x_0, t_0, r_0, -r_0^2) = \{(x, t) | x \in B_t(x_0, r_0), t \in [t_0 - r_0^2, t_0]\}$ (whenever it is defined), we can require $\widetilde{\delta} > 0$, depending on r and $\widetilde{r}_\ell$, so that $\gamma(T_0)$ does not lie in the region $P(x_0, t_0, r_0, -r_0^2)$. Let $\triangle t$ be

maximal such that $\gamma|_{[t_0-\triangle t, t_0]} \subset P(x_0, t_0, r_0, -\triangle t)$ (i.e., $t = t_0 - \triangle t$ is the first time for γ escaping the parabolic region $P(x_0, t_0, r_0, -r_0^2)$). Obviously we may assume that

$$\int_0^{\triangle t} \sqrt{\tau}(R(\gamma(t_0 - \tau), t_0 - \tau) + |\dot{\gamma}(t_0 - \tau)|^2_{g_{ij}(t_0 - \tau)})d\tau < L.$$

If $\triangle t < r_0^2$, it follows from the curvature bound $|Rm| \le r_0^{-2}$ on $P(x_0, t_0, r_0, -r_0^2)$ and the Ricci flow equation that

$$\int_0^{\triangle t} |\dot{\gamma}(t_0 - \tau)|d\tau \ge cr_0$$

for some universal positive constant c. On the other hand, by Cauchy-Schwartz inequality, we have

$$\int_0^{\triangle t} |\dot{\gamma}(t_0 - \tau)|d\tau \le \left(\int_0^{\triangle t} \sqrt{\tau}(R + |\dot{\gamma}|^2)d\tau\right)^{\frac{1}{2}} \cdot \left(\int_0^{\triangle t} \frac{1}{\sqrt{\tau}}d\tau\right)^{\frac{1}{2}}$$

$$\le 2L^{\frac{1}{2}}(\triangle t)^{\frac{1}{4}}$$

which yields

$$(\triangle t)^{\frac{1}{2}} \ge \frac{c^2 r_0^2}{4L}.$$

Thus we always have

$$(\triangle t)^{\frac{1}{2}} \ge \min\left\{r_0, \frac{c^2 r_0^2}{4L}\right\}.$$

Then

$$\int_0^{t_0 - T_0} \sqrt{\tau}(R + |\dot{\gamma}|^2)d\tau \ge (\triangle t)^{\frac{1}{2}} \int_{\triangle t}^{t_0 - T_0} (R + |\dot{\gamma}|^2)d\tau$$

$$\ge \min\left\{r_0, \frac{c^2 r_0^2}{4L}\right\} \int_{\triangle t}^{t_0 - T_0} (R + |\dot{\gamma}|^2)d\tau.$$

By applying Claim 2, we can require the above $\widetilde{\delta}$ further to find $\widetilde{\delta} = \widetilde{\delta}(L, r, \widetilde{r}_\ell) > 0$ so small that as $0 < \delta < \widetilde{\delta}$, there holds

$$\int_{\triangle t}^{t_0 - T_0} (R + |\dot{\gamma}|^2)d\tau \ge L\left(\min\left\{r_0, \frac{c^2 r_0^2}{4L}\right\}\right)^{-1}.$$

Hence we have verified the desired assertion (5.5).

Now choose $L = 100$ in (5.5); then it follows from Claim 1 that there exists $\widetilde{\delta} > 0$, depending on r and $\widetilde{r}_\ell$, such that as each δ-cutoff surgery at the time interval $t \in [(\ell - 1)\varepsilon^2, T]$ has $\delta < \widetilde{\delta}$, every barely admissible curve γ with endpoints (x_0, t_0) and (x, t), where $t \in [(\ell - 1)\varepsilon^2, t_0)$, has

$$L(\gamma) = \int_0^{t_0 - t} \sqrt{\tau}(R(\gamma(\tau), t_0 - \tau) + |\dot{\gamma}(\tau)|^2_{g_{ij}(t_0 - \tau)})d\tau \ge 100,$$

which implies that the reduced distance from (x_0, t_0) to (x, t) satisfies

$$(5.8) \qquad\qquad\qquad l \geq 25\varepsilon^{-1}.$$

We also observe that the absolute value of $l(x_0, \tau)$ is very small as τ closes to zero. We can then apply a maximum principle argument as in Section 7.1 of [**31**] to conclude

$$
\begin{aligned}
l_{\min}(\tau) \\
&= \min\{l(x, \tau)|\ x \text{ lies on the solution manifold at time } t_0 - \tau\} \\
&\leq 2,
\end{aligned}
$$

for $\tau \in (0, t_0 - (\ell - 1)\varepsilon^2]$, because barely admissible curves do not carry a minimum. In particular, there exists a minimizing curve γ of $l_{\min}(t_0 - (\ell - 1)\varepsilon^2)$, defined on $\tau \in [0, t_0 - (\ell - 1)\varepsilon^2]$ with $\gamma(0) = x_0$, such that

$$(5.9) \qquad\qquad\qquad L(\gamma) \leq 2 \cdot (2\sqrt{2}\varepsilon) < 10\varepsilon.$$

Consequently, there exists a point $(\overline{x}, \overline{t})$ on the minimizing curve γ with $\overline{t} \in [(\ell - 1)\varepsilon^2 + \frac{1}{4}\varepsilon^2, (\ell - 1)\varepsilon^2 + \frac{3}{4}\varepsilon^2]$ such that

$$(5.10) \qquad\qquad\qquad R(\overline{x}, \overline{t}) \leq 50\widetilde{r}_\ell^{-2}.$$

Otherwise, we would have

$$
\begin{aligned}
L(\gamma) &\geq \int_{t_0 - (\ell - 1)\varepsilon^2 - \frac{3}{4}\varepsilon^2}^{t_0 - (\ell - 1)\varepsilon^2 - \frac{1}{4}\varepsilon^2} \sqrt{\tau} R(\gamma(\tau), t_0 - \tau) d\tau \\
&\geq 50\widetilde{r}_\ell^{-2} \cdot \frac{2}{3}\left(\frac{1}{2}\varepsilon^2\right)^{\frac{3}{2}} \\
&> 10\varepsilon
\end{aligned}
$$

since $0 < \widetilde{r}_\ell < \varepsilon$; this contradicts (5.9).

Next we want to get a lower bound for the reduced volume of a ball around $\overline{x}$ of radius about $\widetilde{r}_\ell$ at some time-slice slightly before $\overline{t}$. Since the solution satisfies the canonical neighborhood assumption on the time interval $[(\ell - 1)\varepsilon^2, \ell\varepsilon^2)$, it follows from the gradient estimate (5.4) that

$$(5.11) \qquad\qquad\qquad R(x, t) \leq 400\widetilde{r}_\ell^{-2}$$

for those $(x, t) \in P(\overline{x}, \overline{t}, \frac{1}{16}\eta^{-1}\widetilde{r}_\ell, -\frac{1}{64}\eta^{-1}\widetilde{r}_\ell^2)$ for which the solution is defined. And since the points where occur the δ-cutoff surgeries in the time interval $[(\ell-1)\varepsilon^2, \ell\varepsilon^2)$ have their scalar curvature at least $\delta^{-2}\widetilde{r}_\ell^{-2}$, the solution is defined on the whole parabolic region $P(\overline{x}, \overline{t}, \frac{1}{16}\eta^{-1}\widetilde{r}_\ell, -\frac{1}{64}\eta^{-1}\widetilde{r}_\ell^2)$ (this says, this parabolic region is unaffected by surgery). Thus by combining (5.9) and (5.11), the reduced distance from (x_0, t_0) to each point of the ball $B_{\overline{t} - \frac{1}{64}\eta^{-1}\widetilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\widetilde{r}_\ell)$ is uniformly

bounded by some universal constant. Let us define the reduced volume of the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell)$ by

$$\widetilde{V}_{t_0-\bar{t}+\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}\left(B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}\left(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell\right)\right)$$

$$= \int_{B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x},\frac{1}{16}\eta^{-1}\tilde{r}_\ell)} \left(4\pi\left(t_0 - \bar{t} + \frac{1}{64}\eta^{-1}\tilde{r}_\ell^2\right)\right)^{-2}$$

$$\cdot \exp\left(-l\left(q, t_0 - \bar{t} + \frac{1}{64}\eta^{-1}\tilde{r}_\ell^2\right)\right) dV_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(q).$$

Hence by the κ_ℓ-noncollapsing assumption on the time interval $[(\ell-1)\varepsilon^2, \ell\varepsilon^2)$, we conclude that the reduced volume of the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell)$ is bounded from below by a positive constant depending only on κ_ℓ and $\tilde{r}_\ell$.

Finally we want to get a lower bound estimate for the volume of the ball $B_{t_0}(x_0, r_0)$. We have seen that the reduced distance from (x_0, t_0) to each point of the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell)$ is uniformly bounded by some universal constant. Without loss of generality, we may assume $\varepsilon > 0$ is very small. Then it follows from (5.8) that the points in the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell)$ can be connected to (x_0, t_0) by shortest $\mathcal{L}$-geodesics, and all of these $\mathcal{L}$-geodesics are admissible (i.e., they stay in the region unaffected by surgery). The union of all shortest $\mathcal{L}$-geodesics from (x_0, t_0) to the ball $B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell)$, denoted by $CB_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell)$, forms a cone-like subset in space-time with the vertex (x_0, t_0). Denote $B(t)$ by the intersection of $CB_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell)$ with the time-slice at t. The reduced volume of the subset $B(t)$ is defined by

$$\widetilde{V}_{t_0-t}(B(t)) = \int_{B(t)} (4\pi(t_0-t))^{-2}\exp(-l(q, t_0-t))dV_t(q).$$

Since the cone-like subset $CB_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell)$ lies entirely in the region unaffected by surgery, we can apply Perelman's Jacobian comparison [**31**] to conclude that

$$(5.12) \quad \widetilde{V}_{t_0-t}(B(t)) \geq \widetilde{V}_{t_0-\bar{t}+\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}\left(B_{\bar{t}-\frac{1}{64}\eta^{-1}\tilde{r}_\ell^2}\left(\overline{x}, \frac{1}{16}\eta^{-1}\tilde{r}_\ell\right)\right)$$

$$\geq c(\kappa_\ell, \tilde{r}_\ell)$$

for all $t \in [\bar{t} - \frac{1}{64}\eta^{-1}\tilde{r}_\ell^2, t_0]$, where $c(\kappa_\ell, \tilde{r}_\ell)$ is some positive constant depending only on κ_ℓ and $\tilde{r}_\ell$.

Denote by $\xi = r_0^{-1}V_0\ell_{t_0}(B_{t_0}(x_0, r_0))^{\frac{1}{4}}$. Our purpose is to give a positive lower bound for ξ. Without loss of generality, we may assume

$\xi < \frac{1}{4}$, thus $0 < \xi r_0^2 < t_0 - \bar{t} + \frac{1}{64}\eta^{-1}\tilde{r}_\ell^2$. And denote by $\widetilde{B}(t_0 - \xi r_0^2)$ the subset of the points at the time-slice $\{t = t_0 - \xi r_0^2\}$ where every point can be connected to (x_0, t_0) by an admissible shortest $\mathcal{L}$-geodesic. Clearly $B(t_0 - \xi r_0^2) \subset \widetilde{B}(t_0 - \xi r_0^2)$.

Since $r_0 \geq \frac{1}{2\eta} r$ and $\tilde{\delta} = \tilde{\delta}(r, \tilde{r}_\ell, \varepsilon)$ sufficiently small, the region $P(x_0, t_0, r_0, -r_0^2)$ is unaffected by surgery. Then by the exact same argument as used in deriving (3.24) in the proof of Theorem 3.5, we see that there exists a universal positive constant ξ_0 such that as $0 < \xi \leq \xi_0$, there holds

$$(5.13) \qquad \mathcal{L}\exp_{\{|v| \leq \frac{1}{4}\xi^{-\frac{1}{2}}\}}(\xi r_0^2) \subset B_{t_0}(x_0, r_0).$$

The reduced volume $\widetilde{B}(t_0 - \xi r_0^2)$ is given by

$$(5.14)$$
$$\widetilde{V}_{\xi r_0^2}(\widetilde{B}(t_0 - \xi r_0^2))$$
$$= \int_{\widetilde{B}(t_0 - \xi r_0^2)} (4\pi\xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q)$$
$$= \int_{\widetilde{B}(t_0 - \xi r_0^2) \cap \mathcal{L}\exp_{\{|v| \leq \frac{1}{4}\xi^{-\frac{1}{2}}\}}(\xi r_0^2)} (4\pi\xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q)$$
$$+ \int_{\widetilde{B}(t_0 - \xi r_0^2) \setminus \mathcal{L}\exp_{\{|v| \leq \frac{1}{4}\xi^{-\frac{1}{2}}\}}(\xi r_0^2)} (4\pi\xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q).$$

By (5.13), the first term on the RHS of (5.14) can be estimated by

$$(5.15)$$
$$\int_{\widetilde{B}(t_0 - \xi r_0^2) \cap \mathcal{L}\exp_{\{|v| \leq \frac{1}{4}\xi^{-\frac{1}{2}}\}}(\xi r_0^2)} (4\pi\xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q)$$
$$\leq e^{4\xi} \int_{B_{t_0}(x_0, r_0)} (4\pi\xi r_0^2)^{-2} \exp(-l) dV_{t_0}(q)$$
$$\leq e^{4\xi} (4\pi)^{-2} \xi^2.$$

And the second term on the RHS of (5.14) can be estimated by

$$(5.16)$$
$$\int_{\widetilde{B}(t_0 - \xi r_0^2) \setminus \mathcal{L}\exp_{\{|v| \leq \frac{1}{4}\xi^{-\frac{1}{2}}\}}(\xi r_0^2)} (4\pi\xi r_0^2)^{-2} \exp(-l(q, \xi r_0^2)) dV_{t_0 - \xi r_0^2}(q)$$
$$\leq \int_{\{|v| > \frac{1}{4}\xi^{-\frac{1}{2}}\}} (4\pi\tau)^{-2} \exp(-l) J(\tau)|_{\tau=0} dv$$
$$= (4\pi)^{-2} \int_{\{|v| > \frac{1}{4}\xi^{-\frac{1}{2}}\}} \exp(-|v|^2) dv,$$

by using Perelman's Jacobian comparison theorem [**31**] (as deriving (3.30) in the proof of Theorem 3.5). Hence the combination of (5.12), (5.14), (5.15) and (5.16) bounds ξ from below by a positive constant depending only on κ_ℓ and $\widetilde{r}_\ell$.

Therefore we have completed the proof of the lemma.   q.e.d.

Now we can prove the proposition.

Proof of Proposition 5.4. The proof of the proposition is by induction: having constructed our sequences for $1 \leq j \leq \ell$, we make one more step, defining $\widetilde{r}_{\ell+1}$, $\kappa_{\ell+1}$, $\widetilde{\delta}_{\ell+1}$, and redefining $\bar{\widetilde{\delta}}_\ell = \widetilde{\delta}_{\ell+1}$. In view of the previous lemma, we only need to define $\widetilde{r}_{\ell+1}$ and $\widetilde{\delta}_{\ell+1}$.

In Theorem 4.1 we have obtained the canonical neighborhood structure for smooth solutions. When adapting the arguments in the proof of Theorem 4.1 to the present surgical solutions, we will encounter the difficulty of how to take a limit for the surgically modified solutions. The idea to overcome the difficulty consists of two parts. The first part, due to Perelman [**32**], is to choose $\widetilde{\delta}_\ell$ and $\widetilde{\delta}_{\ell+1}$ small enough to push the surgical regions to infinity in space. (This is the reason why we need to redefine $\bar{\widetilde{\delta}}_\ell = \widetilde{\delta}_{\ell+1}$.) The second part is to show that solutions are smooth on some uniform small time intervals (on compact subsets) so that we can apply Hamilton's compactness theorem, since we only have curvature bounds; otherwise Shi's interior derivative estimate may not be applicable. That is just concerned with the question of whether the surgery times accumulate or not.

We now start to prove the proposition by contradiction. Suppose for sequence of positive numbers r^α and $\widetilde{\delta}^{\alpha\beta}$, satisfying $r^\alpha \to 0$ as $\alpha \to \infty$ and $\widetilde{\delta}^{\alpha\beta} \leq \frac{1}{\alpha\beta}(\to 0)$, there exist sequences of solutions $g_{ij}^{\alpha\beta}$ to the Ricci flow with surgery, where each of them has only a finite number of cutoff surgeries and has the given compact four-manifold as initial datum, so that the following two assertions hold:

(i) each δ-cutoff at a time $t \in [(\ell - 1)\varepsilon^2, (\ell + 1)\varepsilon^2]$ satisfies $\delta \leq \widetilde{\delta}^{\alpha\beta}$; and

(ii) the solutions satisfy the statement of the proposition on $[0, \ell\varepsilon^2]$, but violate the canonical neighborhood assumption (with accuracy ε) with $r = r^\alpha$ on $[\ell\varepsilon^2, (\ell + 1)\varepsilon^2]$.

For each solution $g_{ij}^{\alpha\beta}$, we choose $\bar{t}$ (depending on α, β) to be the nearly first time for which the canonical neighborhood assumption (with accuracy ε) is violated. More precisely, we choose $\bar{t} \in [\ell\varepsilon^2, (\ell + 1)\varepsilon^2]$ so that the canonical neighborhood assumption with $r = r^\alpha$ and with accuracy parameter ε is violated at some $(\bar{x}, \bar{t})$; however the canonical neighborhood assumption with accuracy parameter 2ε holds on $t \in [\ell\varepsilon^2, \bar{t}]$. After passing to subsequences, we may assume each $\widetilde{\delta}^{\alpha\beta}$ is less than the $\widetilde{\delta}$ in Lemma 5.5 with $r = r^\alpha$ when α is fixed. Then by Lemma 5.5 we have

uniform κ-noncollapsing for all scales less than ε on $[0,\bar{t}]$ with some $\kappa > 0$ independent of α, β.

Slightly abusing notation, we will often drop the indices α, β.

Let $\widetilde{g}_{ij}^{\alpha\beta}$ be the rescaled solutions along $(\bar{x}, \bar{t})$ with factors $R(\bar{x}, \bar{t})(\geq r^{-2} \to +\infty)$ and shift the times $\bar{t}$ to zero. We hope to take a limit of the rescaled solutions for subsequences of $\alpha, \beta \to \infty$ and show the limit is an ancient κ-solution, which will give the desired contradiction. We divide the following arguments into six steps.

Step 1.   Let $(y, \hat{t})$ be a point on the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ with $\widetilde{R}(y, \hat{t}) \leq A$ $(A \geq 1)$ and $\hat{t} \in [-(\bar{t} - (\ell - 1)\varepsilon^2)R(\bar{x}, \bar{t}), 0]$, then we have estimate

$$(5.17) \qquad\qquad\qquad \widetilde{R}(x, t) \leq 10A$$

for those (x, t) in the parabolic neighborhood $P(y, \hat{t}, \frac{1}{2}\eta^{-1}A^{-\frac{1}{2}}, -\frac{1}{8}\eta^{-1}A^{-1}) \triangleq \{(x', t') \mid x' \in \widetilde{B}_{t'}(y, \frac{1}{2}\eta^{-1}A^{-\frac{1}{2}}), t' \in [\hat{t} - \frac{1}{8}\eta^{-1}A^{-1}, \hat{t}]\}$, for which the rescaled solution is defined.

Indeed, as in the first step of the proof of Theorem 4.1, this follows directly from the gradient estimates (5.4) in the canonical neighborhood assumption with parameter 2ε.

Step 2.   In this step, we will prove three time extending results.

Assertion 1. For arbitrarily fixed α, $0 < A < +\infty$, $1 \leq C < +\infty$ and $0 \leq B < \frac{1}{2}\varepsilon^2(r^\alpha)^{-2} - \frac{1}{8}\eta^{-1}C^{-1}$, there is a $\beta_0 = \beta_0(\varepsilon, A, B, C)$ (independent of α) such that if $\beta \geq \beta_0$ and the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ on the ball $\widetilde{B}_0(\bar{x}, A)$ is defined on a time interval $[-b, 0]$ with $0 \leq b \leq B$ and the scalar curvature satisfies

$$\widetilde{R}(x, t) \leq C, \quad \text{on } \widetilde{B}_0(\bar{x}, A) \times [-b, 0],$$

then the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ on the ball $\widetilde{B}_0(\bar{x}, A)$ is also defined on the extended time interval $[-b - \frac{1}{8}\eta^{-1}C^{-1}, 0]$.

Before the proof, we need a simple **observation**: once a space point in the Ricci flow with surgery is removed by surgery at some time, then it never appears later time; if a space point at some time t can not be defined before the time t, then either the point lies in a gluing cap of the surgery at time t or the time t is the initial time of the Ricci flow.

Proof of Assertion 1. Firstly we claim that there exists $\beta_0 = \beta_0(\varepsilon, A, B, C)$ such that as $\beta \geq \beta_0$, the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ on the ball $\widetilde{B}_0(\bar{x}, A)$ can be defined before the time $-b$ (i.e., there are no surgeries interfering in $\widetilde{B}_0(\bar{x}, A) \times [-b - \epsilon', -b]$ for some $\epsilon' > 0$).

We argue by contradiction. Suppose not, then there is some point $\tilde{x} \in \widetilde{B}_0(\bar{x}, A)$ such that the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ at $\tilde{x}$ can not be defined

before the time $-b$. By the above observation, there is a surgery at the time $-b$ such that the point $\tilde{x}$ lies in the instant gluing cap.

Let $\tilde{h}$ $(= R(\bar{x}, \bar{t})^{\frac{1}{2}} h)$ be the cut-off radius at the time $-b$ for the rescaled solution. Clearly, there is a universal constant D such that

$$D^{-1}\tilde{h} \leq \widetilde{R}(\tilde{x}, -b)^{-\frac{1}{2}} \leq D\tilde{h}.$$

By Lemma 5.3 and looking at the rescaled solution at the time $-b$, the gluing cap and the adjacent δ-neck, of radius $\tilde{h}$, constitute a $\left(\widetilde{\delta^{\alpha\beta}}\right)^{\frac{1}{2}}$-cap $\mathcal{K}$. For any fixed small positive constant δ' (much smaller than ε), we see

$$\widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\widetilde{R}(\tilde{x}, -b)^{-\frac{1}{2}}) \subset \mathcal{K}$$

as β large enough. We first verify the following

Claim 1. For any small constants $0 < \tilde{\theta} < \frac{3}{2}$, $\delta' > 0$, there exists a $\beta(\delta', \varepsilon, \tilde{\theta}) > 0$ such that as $\beta \geq \beta(\delta', \varepsilon, \tilde{\theta})$, we have

(i) the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ over $\widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\tilde{h})$ is defined on the time interval $[-b, 0] \cap [-b, -b + (\frac{3}{2} - \tilde{\theta})\tilde{h}^2]$;

(ii) the ball $\widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\tilde{h})$ in the $\left(\widetilde{\delta^{\alpha\beta}}\right)^{\frac{1}{2}}$-cap $\mathcal{K}$ evolved by the Ricci flow on the time interval $[-b, 0] \cap [-b, -b + (\frac{3}{2} - \tilde{\theta})\tilde{h}^2]$ is, after scaling with factor $\tilde{h}^{-2}$, δ'-close (in $C^{[\delta'^{-1}]}$ topology) to the corresponding subset of the standard solution.

This claim is somewhat known in the first claim in the proof of Lemma 5.5. Indeed, suppose there is a surgery at some time $\widetilde{\tilde{t}} \in [-b, 0] \cap (-b, -b + (\frac{3}{2} - \tilde{\theta})\tilde{h}^2]$ which removes some point $\widetilde{\tilde{x}} \in \widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\tilde{h})$. We assume $\widetilde{\tilde{t}} \in (-b, 0]$ to be the first time with that property.

Then by the proof of the first claim in Lemma 5.5, there is a $\bar{\delta} = \bar{\delta}(\delta', \varepsilon, \tilde{\theta})$ such that if $\widetilde{\delta^{\alpha\beta}} < \bar{\delta}$, then the ball $\widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\tilde{h})$ in the $\left(\widetilde{\delta^{\alpha\beta}}\right)^{\frac{1}{2}}$-cap $\mathcal{K}$ evolved by the Ricci flow on the time interval $[-b, \widetilde{\tilde{t}})$ is, after scaling with factor $\tilde{h}^{-2}$, δ'-close to the corresponding subset of the standard solution. Note that the metrics for times in $[-b, \widetilde{\tilde{t}})$ on $\widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\tilde{h})$ are equivalent. By the proof of the first claim in Lemma 5.5, the solution on $\widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\tilde{h})$ keeps looking like a cap for $t \in [-b, \widetilde{\tilde{t}})$. On the other hand, by definition, the surgery is always performed along the middle three-sphere of a δ-neck with $\delta < \widetilde{\delta^{\alpha\beta}}$. Then as β large, all the points in $\widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\tilde{h})$ are removed (as a part of a capped horn) at the time $\widetilde{\tilde{t}}$. But $\tilde{x}$ (near the tip of the cap) exists past the time $\widetilde{\tilde{t}}$. This is a contradiction. Hence we have proved that $\widetilde{B}_{(-b)}(\tilde{x}, (\delta')^{-1}\tilde{h})$ is defined on the time interval $[-b, 0] \cap [-b, -b + (\frac{3}{2} - \tilde{\theta})\tilde{h}^2]$.

The δ'-closeness of the solution on $\widetilde{B}_{(-b)}(\tilde{x},(\delta')^{-1}h)\times([-b,0]\cap[-b,-b$ $+(\frac{3}{2}-\tilde{\theta})\tilde{h}^2])$ with the corresponding subset of the standard solution follows by the uniqueness theorem and the canonical neighborhood assumption with parameter 2ε as in the proof of the first claim in Lemma 5.5. Then we have proved Claim 1.

We next verify the following

Claim 2. There is $\tilde{\theta}=\tilde{\theta}(CB)$, $0<\tilde{\theta}<\frac{3}{2}$, such that $b\leq(\frac{3}{2}-\tilde{\theta})\tilde{h}^2$ as β large.

Note from Theorem A.1 in Appendix, that there is a universal constant $D'>0$ such that the standard solution (of dimension four) satisfies the following curvature estimate

$$R(y,s)\geq\frac{2D'}{\frac{3}{2}-s}.$$

We choose $\tilde{\theta}=3D'/2(2D'+2CB)$. Then as β large enough, the rescaled solution satisfies

(5.18) $$\widetilde{R}(x,t)\geq\frac{D'}{\frac{3}{2}-(t+b)\tilde{h}^{-2}}\tilde{h}^{-2}$$

on $\widetilde{B}_{(-b)}(\tilde{x},(\delta')^{-1}\tilde{h})\times([-b,0]\cap[-b,-b+(\frac{3}{2}-\tilde{\theta})\tilde{h}^2])$.

Suppose $b\geq(\frac{3}{2}-\tilde{\theta})\tilde{h}^2$. Then by combining with the assumption $\widetilde{R}(\tilde{x},t)\leq C$ for $t=(\frac{3}{2}-\tilde{\theta})\tilde{h}^2-b$, we have

$$C\geq\frac{D'}{\frac{3}{2}-(t+b)\tilde{h}^{-2}}\tilde{h}^{-2},$$

and then

$$\tilde{\theta}\geq\frac{\frac{3D'}{2CB}}{1+\frac{D'}{CB}}.$$

This is a contradiction. Hence we have proved Claim 2.

The combination of the above two claims shows that there is a positive constant $0<\tilde{\theta}=\tilde{\theta}(CB)<\frac{3}{2}$ such that for any $\delta'>0$, there is a positive $\beta(\delta',\varepsilon,\tilde{\theta})$ such that as $\beta\geq\beta(\delta',\varepsilon,\tilde{\theta})$, we have $b\leq(\frac{3}{2}-\tilde{\theta})\tilde{h}^2$ and the rescaled solution in the ball $\widetilde{B}_{(-b)}(\tilde{x},(\delta')^{-1}\tilde{h})$ on the time interval $[-b,0]$ is, after scaling with factor $\tilde{h}^{-2}$, δ'-close (in $C^{[(\delta')^{-1}]}$ topology) to the corresponding subset of the standard solution.

By (5.18) and the assumption $\widetilde{R}\leq C$ on $\widetilde{B}_0(\bar{x},A)\times[-b,0]$, we know that the cut-off radius $\tilde{h}$ at the time $-b$ for the rescaled solution satisfies

$$\tilde{h}\geq\sqrt{\frac{2D'}{3C}}.$$

Let $\delta'>0$ be much smaller than ε and $\min\{A^{-1},A\}$. Since $\tilde{d}_0(\tilde{x},\bar{x})\leq A$, it follows that there is constant $C(\tilde{\theta})$ depending only on $\tilde{\theta}$ such that

$\tilde{d}_{(-b)}(\tilde{x}, \bar{x}) \leq C(\tilde{\theta})A \ll (\delta')^{-1}\tilde{h}$. We now apply Corollary A.2 in Appendix with the accuracy parameter $\varepsilon/2$. Let $C(\varepsilon/2)$ be the positive constant in Corollary A.2. Without loss of generality, we may assume the positive constant $C_1(\varepsilon)$ in the canonical neighborhood assumption is larger than $4C(\varepsilon/2)$. As $\delta' > 0$ is much smaller than ε and $\min\{A^{-1}, A\}$, the point $\bar{x}$ at the time $\bar{t}$ has a neighborhood which is either a $\frac{3}{4}\varepsilon$-cap or a $\frac{3}{4}\varepsilon$-neck.

Since the canonical neighborhood assumption with accuracy parameter ε is violated at $(\bar{x}, \bar{t})$, the neighborhood of the point $\bar{x}$ at the new time zero for the rescaled solution must be a $\frac{3}{4}\varepsilon$-neck. By Corollary A.2 (b), we know the neighborhood is the slice at the time zero of the parabolic neighborhood

$$P\left(\bar{x}, 0, \frac{4}{3}\varepsilon^{-1}\widetilde{R}(\bar{x}, 0)^{-\frac{1}{2}}, -\min\{\widetilde{R}(\bar{x}, 0)^{-1}, b\}\right)$$

(with $\widetilde{R}(\bar{x}, 0) = 1$) which is $\frac{3}{4}\varepsilon$-close (in $C^{[\frac{4}{3}\varepsilon^{-1}]}$ topology) to the corresponding subset of the evolving standard cylinder $\mathbb{S}^3 \times \mathbb{R}$ over the time interval $[-\min\{b, 1\}, 0]$ with scalar curvature 1 at the time zero. If $b \geq 1$, the $\frac{3}{4}\varepsilon$-neck is strong, which is a contradiction. While if $b < 1$, the $\frac{3}{4}\varepsilon$-neck at time $-b$ is contained in the union of the gluing cap and the adjacent δ-neck where the δ-cutoff surgery was taken. Since ε is small (say $\varepsilon < 1/100$), it is clear that the point $\bar{x}$ at time $-b$ is the center of an ε-neck which is entirely contained in the adjacent δ-neck. By the remark after Lemma 5.2, the adjacent δ-neck approximates an ancient κ-solution. This implies the point $\bar{x}$ at the time $\bar{t}$ has a strong ε-neck, which is also a contradiction.

Hence we have proved that there exists $\beta_0 = \beta_0(\varepsilon, A, B, C)$ such that as $\beta \geq \beta_0$, the rescaled solution on the ball $\widetilde{B}_0(\bar{x}, A)$ can be defined before the time $-b$.

Let $[t_A^{\alpha\beta}, 0] \supset [-b, 0]$ be the largest time interval so that the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ can be defined on $\widetilde{B}_0(\bar{x}, A) \times [t_A^{\alpha\beta}, 0]$. We finally claim that $t_A^{\alpha\beta} \leq -b - \frac{1}{8}\eta^{-1}C^{-1}$ as β large enough.

Indeed, suppose not; by the gradient estimates as in Step 1, we have the curvature estimate

$$\widetilde{R}(x, t) \leq 10C$$

on $\widetilde{B}_0(\bar{x}, A) \times [t_A^{\alpha\beta}, -b]$. Hence we have the curvature estimate

$$\widetilde{R}(x, t) \leq 10C$$

on $\widetilde{B}_0(\bar{x}, A) \times [t_A^{\alpha\beta}, 0]$. By the above argument there is a $\beta_0 = \beta_0(\varepsilon, A, B + \frac{1}{8}\eta^{-1}C^{-1}, 10C)$ such that as $\beta \geq \beta_0$, the solution in the ball $\widetilde{B}_0(\bar{x}, A)$ can be defined before the time $t_A^{\alpha\beta}$. This is a contradiction.

Therefore we have proved Assertion 1.

Assertion 2. For arbitrarily fixed α, $0 < A < +\infty$, $1 \leq C < +\infty$ and $0 < B < \frac{1}{2}\varepsilon^2(r^\alpha)^{-2} - \frac{1}{50}\eta^{-1}$, there is a $\beta_0 = \beta_0(\varepsilon, A, B, C)$ (independent of α) such that if $\beta \geq \beta_0$ and the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ on the ball $\widetilde{B}_0(\bar{x}, A)$ is defined on a time interval $[-b + \epsilon', 0]$ with $0 < b \leq B$ and $0 < \epsilon' < \frac{1}{50}\eta^{-1}$ and the scalar curvature satisfies

$$\widetilde{R}(x, t) \leq C \quad \text{on} \quad \widetilde{B}_0(\bar{x}, A) \times [-b + \epsilon', 0],$$

and there is a point $y \in \widetilde{B}_0(\bar{x}, A)$ such that $\widetilde{R}(y, -b + \epsilon') \leq \frac{3}{2}$, then the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ at y is also defined on the extended time interval $[-b - \frac{1}{50}\eta^{-1}, 0]$ and satisfies the estimate

$$\widetilde{R}(y, t) \leq 15$$

for $t \in [-b - \frac{1}{50}\eta^{-1}, -b + \epsilon']$.

Proof of Assertion 2. We imitate the proof of Assertion 1. If the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ at y can not be defined for some time in $[-b - \frac{1}{50}\eta^{-1}, -b + \epsilon')$, then there is a surgery at some time $\tilde{\tilde{t}} \in [-b - \frac{1}{50}\eta^{-1}, -b + \epsilon']$ such that y lies in the instant gluing cap. Let $\tilde{h}$ $(= R(\bar{x}, \bar{t})^{\frac{1}{2}}h)$ be the cutoff radius at the time $\tilde{\tilde{t}}$ for the rescaled solution. Clearly, there is a universal constant $D > 1$ such that $D^{-1}\tilde{h} \leq \widetilde{R}(y, \tilde{\tilde{t}})^{-\frac{1}{2}} \leq D\tilde{h}$. By the gradient estimates as in Step 1, the cutoff radius satisfies

$$\tilde{h} \geq D^{-1}15^{-\frac{1}{2}}.$$

As in Claim 1 (i) in the proof of Assertion 1, for any small constants $0 < \tilde{\theta} < \frac{3}{2}$, $\delta' > 0$, there exists a $\beta(\delta', \varepsilon, \tilde{\theta}) > 0$ such that as $\beta \geq \beta(\delta', \varepsilon, \tilde{\theta})$, there is no surgery interfering in $\widetilde{B}_{\tilde{\tilde{t}}}(y, (\delta')^{-1}\tilde{h}) \times ([\tilde{\tilde{t}}, (\frac{3}{2} - \tilde{\theta})\tilde{h}^2 + \tilde{\tilde{t}}] \cap (\tilde{\tilde{t}}, 0])$. Without loss of generality, we may assume that the universal constant η is much larger than D. Then we have $(\frac{3}{2} - \tilde{\theta})\tilde{h}^2 + \tilde{\tilde{t}} > -b + \frac{1}{50}\eta^{-1}$. As in Claim 2, we can use the curvature bound assumption to choose $\tilde{\theta} = \tilde{\theta}(B, C)$ such that $(\frac{3}{2} - \tilde{\theta})\tilde{h}^2 + \tilde{\tilde{t}} \geq 0$; otherwise

$$C \geq \frac{D'}{\tilde{\theta}\tilde{h}^2}$$

for some universal constant D', and

$$|\tilde{\tilde{t}} + b| \leq \frac{1}{50}\eta^{-1},$$

which implies

$$\tilde{\theta} \geq \frac{\frac{3D'}{2C(B + \frac{1}{50}\eta^{-1})}}{1 + \frac{D'}{C(B + \frac{1}{50}\eta^{-1})}}.$$

This is a contradiction if we choose $\tilde{\theta} = 3D'/2(2D' + 2C(B + \frac{1}{50}\eta^{-1}))$.

So there is a positive constant $0 < \tilde{\theta} = \tilde{\theta}(B, C) < \frac{3}{2}$ such that for any $\delta' > 0$, there is a positive $\beta(\delta', \varepsilon, \tilde{\theta})$ such that as $\beta \geq \beta(\delta', \varepsilon, \tilde{\theta})$, we have $-\tilde{\tilde{t}} \leq (\frac{3}{2} - \tilde{\theta})\tilde{h}^2$ and the solution in the ball $\widetilde{B}_{\tilde{\tilde{t}}}(\tilde{x}, (\delta')^{-1}\tilde{h})$ on the time interval $[\tilde{\tilde{t}}, 0]$ is, after scaling with factor $\tilde{h}^{-2}$, δ'-close (in $C^{[\delta'^{-1}]}$ topology) to the corresponding subset of the standard solution.

Then exactly as in the proof of Assertion 1, by using the canonical neighborhood structure of the standard solution in Corollary A.2, this gives the desired contradiction with the hypothesis that the canonical neighborhood assumption with accuracy parameter ε is violated at $(\bar{x}, \bar{t})$, as β sufficiently large.

The curvature estimate at the point y follows from Step 1. Therefore we complete the proof of Assertion 2.   q.e.d.

Note that the standard solution satisfies $R(x_1, t) \leq D'' R(x_2, t)$ for any $t \in [0, \frac{1}{2}]$ and any two points x_1, x_2, where $D'' \geq 1$ is a universal constant.

Assertion 3. For arbitrarily fixed α, $0 < A < +\infty$, $1 \leq C < +\infty$, there is a $\beta_0 = \beta_0(\varepsilon, AC^{\frac{1}{2}})$ such that if any point (y_0, t_0) with $0 \leq -t_0 < \frac{1}{2}\varepsilon^2(r^\alpha)^{-2} - \frac{1}{8}\eta^{-1}C^{-1}$ of the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ for $\beta \geq \beta_0$ satisfies $\widetilde{R}(y_0, t_0) \leq C$, then either the rescaled solution at y_0 can be defined at least on $[t_0 - \frac{1}{16}\eta^{-1}C^{-1}, t_0]$ and the rescaled scalar curvature satisfies

$$\widetilde{R}(y_0, t) \leq 10C \quad \text{for } t \in \left[t_0 - \frac{1}{16}\eta^{-1}C^{-1}, t_0\right],$$

or we have

$$\widetilde{R}(x_1, t_0) \leq 2D''\widetilde{R}(x_2, t_0)$$

for any two points $x_1, x_2 \in \widetilde{B}_{t_0}(y_0, A)$, where D'' is the above universal constant.

Proof of Assertion 3. Suppose the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$ at y_0 can not be defined for some $t \in [t_0 - \frac{1}{16}\eta^{-1}C^{-1}, t_0)$; then there is a surgery at some time $\tilde{t} \in [t_0 - \frac{1}{16}\eta^{-1}C^{-1}, t_0]$ such that y_0 lies in the instant gluing cap. Let $\tilde{h}$ $(= R(\bar{x}, \bar{t})^{\frac{1}{2}}h)$ be the cutoff radius at the time $\tilde{t}$ for the rescaled solution $\widetilde{g}_{ij}^{\alpha\beta}$. By the gradient estimates as in Step 1, the cutoff radius satisfies

$$\tilde{h} \geq D^{-1}10^{-\frac{1}{2}}C^{-\frac{1}{2}},$$

where D is the universal constant in the proof of Assertion 1. Since we assume η is suitably larger than D as before, we have $\frac{1}{2}\tilde{h}^2 + \tilde{t} > t_0$. As in Claim 1 (ii) in Assertion 1, for arbitrarily small $\delta' > 0$, we know that as β large enough the rescaled solution on the ball $\widetilde{B}_{\tilde{t}}(y_0, (\delta')^{-1}\tilde{h})$ on the time interval $[\tilde{t}, t_0]$ is, after scaling with factor $\tilde{h}^{-2}$, δ'-close (in $C^{[(\delta')^{-1}]}$ topology) to the corresponding subset of the standard solution. Since

$(\delta')^{-1}\tilde{h} \gg A$ as β large enough, Assertion 3 follows from the curvature estimate of standard solution in the time interval $[0, \frac{1}{2}]$.   q.e.d.

Step 3.   For any subsequence (α_m, β_m) of (α, β) with $r^{\alpha_m} \to 0$ and $\delta^{\alpha_m \beta_m} \to 0$ as $m \to \infty$, we next argue as in the second step of the proof of Theorem 4.1 to show that the curvatures of the rescaled solutions $\tilde{g}^{\alpha_m \beta_m}$ at new times zero (after shifting) stay uniformly bounded at bounded distances from $\bar{x}$ for all sufficiently large m. More precisely, we will prove the following assertion:

Assertion 4. Given any subsequence of the rescaled solutions $\tilde{g}_{ij}^{\alpha_m \beta_m}$ with $r^{\alpha_m} \to 0$ and $\delta^{\alpha_m \beta_m} \to 0$ as $m \to \infty$, for any $L > 0$, there are constants $C(L) > 0$ and $m(L)$ such that the rescaled solutions $\tilde{g}_{ij}^{\alpha_m \beta_m}$ satisfy

   (i) $\tilde{R}(x, 0) \leq C(L)$ for all points x with $\tilde{d}_0(x, \bar{x}) \leq L$ and all $m \geq 1$;
   (ii) the rescaled solutions over the ball $\tilde{B}_0(\bar{x}, L)$ are defined at least on the time interval $[-\frac{1}{16}\eta^{-1}C(L)^{-1}, 0]$ for all $m \geq m(L)$.

Proof of Assertion 4. For all $\rho > 0$, set

$$M(\rho) = \sup\left\{ \tilde{R}(x, 0) \mid m \geq 1 \text{ and } \tilde{d}_0(x, \bar{x}) \leq \rho \right.$$

$$\left. \text{in the rescaled solutions } \tilde{g}_{ij}^{\alpha_m \beta_m} \right\}$$

and

$$\rho_0 = \sup\{\rho > 0 \mid M(\rho) < +\infty\}.$$

Note that the estimate (5.17) implies that $\rho_0 > 0$. For (i), it suffices to prove $\rho_0 = +\infty$.

We argue by contradiction. Suppose $\rho_0 < +\infty$. Then there are a sequence of points y in the rescaled solutions $\tilde{g}_{ij}^{\alpha_m \beta_m}$ with $\tilde{d}_0(\bar{x}, y) \to \rho_0 < +\infty$ and $\tilde{R}(y, 0) \to +\infty$. Denote by γ a minimizing geodesic segment from $\bar{x}$ to y and denote by $\tilde{B}_0(\bar{x}, \rho_0)$ the geodesic open ball centered at $\bar{x}$ of radius ρ_0 on the rescaled solution $\tilde{g}_{ij}^{\alpha_m \beta_m}$.

First, we claim that for any $0 < \rho < \rho_0$ with ρ near ρ_0, the rescaled solutions on the balls $\tilde{B}_0(\bar{x}, \rho)$ are defined on the time interval $[-\frac{1}{16}\eta^{-1}M(\rho)^{-1}, 0]$ for all large m. Indeed, this follows from Assertion 3 or Assertion 1. For the later purpose in Step 6, we now present an argument by using Assertion 3. If the claim is not true, then there is a surgery at some time $\tilde{t} \in [-\frac{1}{16}\eta^{-1}M(\rho)^{-1}, 0]$ such that some point $\tilde{y} \in \tilde{B}_0(\bar{x}, \rho)$ lies in the instant gluing cap. We can choose sufficiently small $\delta' > 0$ such that $2\rho_0 < (\delta')^{-\frac{1}{2}}\tilde{h}$, where $\tilde{h} \geq D^{-1}20^{-\frac{1}{2}}M(\rho)^{-\frac{1}{2}}$ are the cutoff radius of the rescaled solutions at $\tilde{t}$. By applying Assertion 3 with $(\tilde{y}, 0) = (y_0, t_0)$, we see that there is a $m(\rho_0, M(\rho)) > 0$ such that as $m \geq m(\rho_0, M(\rho))$,

$$\widetilde{R}(x, 0) \leq 2D''$$

for all $x \in \widetilde{B}_0(\bar{x}, \rho)$. This is a contradiction as $\rho \to \rho_0$.

Since for each fixed $0 < \rho < \rho_0$ with ρ near ρ_0, the rescaled solutions on the ball $\widetilde{B}_0(\bar{x}, \rho)$ are defined on the time interval $[-\frac{1}{16}\eta^{-1}M(\rho)^{-1}, 0]$ for all large m, by Step 1 and Shi's derivative estimate, we know that the covariant derivatives and higher order derivatives of the curvatures on $\widetilde{B}_0(\bar{x}, \rho - \frac{(\rho_0 - \rho)}{2}) \times [-\frac{1}{32}\eta^{-1}M(\rho)^{-1}, 0]$ are also uniformly bounded.

By the uniform κ-noncollapsing and the virtue of Hamilton's compactness theorem 16.1 in [20] (see [3] for the details on generalizing Hamilton's compactness theorem to finite balls), after passing to a subsequence, we can assume that the marked sequence $(\widetilde{B}_0(\bar{x}, \rho_0), \widetilde{g}_{ij}^{\alpha_m \beta_m}, \bar{x})$ converges in C^∞_{loc} topology to a marked (noncomplete) manifold $(B_\infty, \widetilde{g}_{ij}^\infty, \bar{x})$ and the geodesic segments γ converge to a geodesic segment (missing an endpoint) $\gamma_\infty \subset B_\infty$ emanating from $\bar{x}$.

Clearly, the limit has restricted isotropic curvature pinching (2.4) by the pinching assumption. Consider a tubular neighborhood along γ_∞ defined by

$$V = \bigcup_{q_0 \in \gamma_\infty} B_\infty(q_0, 4\pi(\widetilde{R}_\infty(q_0))^{-\frac{1}{2}}),$$

where $\widetilde{R}_\infty$ denotes the scalar curvature of the limit and $B_\infty(q_0, 4\pi(\widetilde{R}_\infty(q_0))^{-\frac{1}{2}})$ is the ball centered at $q_0 \in B_\infty$ with the radius $4\pi(\widetilde{R}_\infty(q_0))^{-\frac{1}{2}}$. Let $\bar{B}_\infty$ denote the completion of $(B_\infty, \widetilde{g}_{ij}^\infty)$, and $y_\infty \in \bar{B}_\infty$ the limit point of γ_∞. Exactly as in the second step of the proof of Theorem 4.1, it follows from the canonical neighborhood assumption with accuracy parameter 2ε that the limiting metric $\widetilde{g}_{ij}^\infty$ is cylindrical at any point $q_0 \in \gamma_\infty$ which is sufficiently close to y_∞ and then the metric space $\bar{V} = V \cup \{y_\infty\}$ by adding the point y_∞ has nonnegative curvature in Alexandrov sense. Consequently we have a four-dimensional non-flat tangent cone $C_{y_\infty}\bar{V}$ at y_∞ which is a metric cone with aperture $\leq 20\varepsilon$.

On the other hand, note that by the canonical neighborhood assumption, the canonical 2ε-neck neighborhoods are strong. Thus at each point $q \in V$ near y_∞, the limiting metric $\widetilde{g}_{ij}^\infty$ actually exists on the whole parabolic neighborhood

$$V \bigcap P\left(q, 0, \frac{1}{3}\eta^{-1}(\widetilde{R}_\infty(q))^{-\frac{1}{2}}, -\frac{1}{10}\eta^{-1}(\widetilde{R}_\infty(q))^{-1}\right),$$

and is a smooth solution of the Ricci flow there. Pick $z \in C_{y_\infty}\bar{V}$ with distance one from the vertex y_∞ and nonflat around z. By definition the ball $B(z, \frac{1}{2}) \subset C_{y_\infty}\bar{V}$ is the Gromov-Hausdorff convergent limit of the scalings of a sequence of balls $B_\infty(z_k, \sigma_k)(\subset (V, \widetilde{g}_{ij}^\infty))$ where $\sigma_k \to 0$. Since the estimate (5.17) survives on $(V, \widetilde{g}_{ij}^\infty)$ for all $A < +\infty$, and the tangent cone is four-dimensional and nonflat around z, we see that this convergence is actually in C^∞_{loc} topology and over some ancient time

interval. Since the limiting $B_\infty(z, \frac{1}{2})(\subset C_{y_\infty} \bar{V})$ is a piece of nonnegatively (operator) curved nonflat metric cone, we get a contradiction with Hamilton's strong maximum principle [**16**] as before. So we have proved $\rho_0 = \infty$. This proves (i).

By the same proof of Assertion 1 in Step 2, we can further show that for any L, the rescaled solutions on the balls $\tilde{B}_0(\bar{x}, L)$ are defined at least on the time interval $[-\frac{1}{16}\eta^{-1}C(L)^{-1}, 0]$ for all sufficiently large m. This proves (ii).   q.e.d.

Step 4.   For any subsequence (α_m, β_m) of (α, β) with $r^{\alpha_m} \to 0$ and $\tilde{\delta}^{\alpha_m \beta_m} \to 0$ as $m \to \infty$, by Step 3, the κ-noncollapsing and Hamilton's compactness theorem, we can extract a C_{loc}^∞ convergent subsequence of $\tilde{g}_{ij}^{\alpha_m \beta_m}$ over some space time open subsets containing $t = 0$. We now want to show **any** such limit has bounded curvature at $t = 0$. We prove by contradiction. Suppose not: then there is a sequence of points z_k divergent to infinity in the limiting metric at time zero with curvature divergent to infinity. Since the curvature at z_k is large (comparable to one), z_k has a canonical neighborhood which is a 2ε-cap or strong 2ε-neck. Note that the boundary of 2ε-cap lies in some 2ε-neck. So we get a sequence of 2ε-necks with radius going to zero. Note also that the limit has nonnegative sectional curvature. Without loss of the generality, we may assume $2\varepsilon < \varepsilon_0$, where ε_0 is the positive constant in Proposition 2.2. Thus arrives a contradiction with Proposition 2.2.

Step 5.   In this step, we will choose some subsequence (α_m, β_m) of (α, β) so that we can extract a complete smooth limit on a time interval $[-a, 0]$ for some $a > 0$ from the rescaled solutions $\tilde{g}_{ij}^{\alpha_m \beta_m}$ of the Ricci flow with surgery.

Choose $\alpha_m, \beta_m \to \infty$ so that $r^{\alpha_m} \to 0$, $\tilde{\delta}^{\alpha_m \beta_m} \to 0$, and Assertions 1, 2, 3 hold with $\alpha = \alpha_m, \beta = \beta_m$ for all $A \in \{p/q \mid p, q = 1, 2 \ldots, m\}$, and $B, C \in \{1, 2, \ldots, m\}$. By Step 3, we may assume the rescaled solutions $\tilde{g}_{ij}^{\alpha_m \beta_m}$ converge in C_{loc}^∞ topology at the time $t = 0$. Since the curvature of the limit at $t = 0$ is bounded by Step 4, it follows from Assertion 1 in Step 2 and the choice of the subsequence (α_m, β_m) that the limiting $(M_\infty, \tilde{g}_{ij}^\infty(\cdot, t))$ is defined at least on a backward time interval $[-a, 0]$ for some positive constant a and is a smooth solution to the Ricci flow there.

Step 6.   We further want to extend the limit of Step 5 backward in time to infinity to get an ancient κ-solution. Let $\tilde{g}_{ij}^{\alpha_m \beta_m}$ be the convergent sequence obtained in the above Step 5.

Denote by

$$t_{\max} = \sup \Big\{ \ t' |\text{ we can take a smooth limit on } (-t', 0] \text{ (with bounded}$$
$$\text{curvature at each time slice) from a subsequence of}$$
$$\text{the rescaled solutions } \widetilde{g}_{ij}^{\alpha_m \beta_m} \Big\}.$$

We first claim that there is a subsequence of the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m \beta_m}$ which converges in C_{loc}^{∞} topology to a smooth limit $(M_{\infty}, \widetilde{g}_{ij}^{\infty}(\cdot, t))$ on the maximal time interval $(-t_{\max}, 0]$.

Indeed, let t_k be a sequence of positive numbers such that $t_k \to t_{\max}$ and there exist smooth limits $(M_{\infty}, \widetilde{g}_k^{\infty}(\cdot, t))$ defined on $(-t_k, 0]$. For each k, the limit has nonnegative curvature operator and has bounded curvature at each time slice. Moreover, by the gradient estimate in canonical neighborhood assumption with accuracy parameter 2ε, the limit has bounded curvature on each subinterval $[-b, 0] \subset (-t_k, 0]$. Denote by $\widetilde{Q}$ the scalar curvature upper bound of the limit at time zero ($\widetilde{Q}$ independent of k). Then we can apply Li-Yau-Hamilton inequality [18] to get

$$\widetilde{R}_k^{\infty}(x, t) \leq \frac{t_k}{t + t_k} \widetilde{Q},$$

where $\widetilde{R}_k^{\infty}(x, t)$ are the scalar curvatures of the limits $(M_{\infty}, \widetilde{g}_k^{\infty}(\cdot, t))$. Hence by the definition of convergence and the above curvature estimates, we can find a subsequence of the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m \beta_m}$ which converges in C_{loc}^{∞} topology to a smooth limit $(M_{\infty}, \widetilde{g}_{ij}^{\infty}(\cdot, t))$ on the maximal time interval $(-t_{\max}, 0]$.

We need to show $-t_{\max} = -\infty$. Suppose $-t_{\max} > -\infty$, then there are only the following two possibilities: either

(1) The curvature of the limiting solution $(M_{\infty}, \widetilde{g}_{ij}^{\infty}(\cdot, t))$ becomes unbounded as $t \searrow -t_{\max}$; or

(2) For each small constant $\theta > 0$ and each large integer $m_0 > 0$, there is some $m \geq m_0$ such that the rescaled solution $\widetilde{g}_{ij}^{\alpha_m \beta_m}$ has a surgery time $T_m \in [-t_{\max} - \theta, 0]$ and a surgery point x_m lying in a gluing cap at the times T_m so that $d_{T_m}^2(x, \bar{x})$ is uniformly bounded from above by a constant independent of θ and m_0.

We next claim that the possibility (1) always occurs. Suppose not, then the curvature of the limiting solution $(M_{\infty}, \widetilde{g}_{ij}^{\infty}(\cdot, t))$ is uniformly bounded by (some positive constant) $\hat{C}$ on $(-t_{\max}, 0]$. In particular, for any $A > 0$, there is a sufficiently large integer $m_1 > 0$ such that any rescaled solution $\widetilde{g}_{ij}^{\alpha_m \beta_m}$ with $m \geq m_1$ on the geodesic ball $\widetilde{B}_0(\bar{x}, A)$ is defined on the time interval $[-t_{\max} + \frac{1}{50}\eta^{-1}\hat{C}^{-1}, 0]$ and its scalar curvature is bounded by $2\hat{C}$ there. (Here, without loss of generality, we may assume that the upper bound $\hat{C}$ is so large that $-t_{\max} + \frac{1}{50}\eta^{-1}\hat{C}^{-1} <$

0.) By Assertion 1 in Step 2, as m large enough, the rescaled solution $\widetilde{g}_{ij}^{\alpha_m\beta_m}$ over $\widetilde{B}_0(\bar{x}, A)$ can be defined on the extended time interval $[-t_{\max} - \frac{1}{50}\eta^{-1}\hat{C}^{-1}, 0]$ and have the scalar curvature $\widetilde{R} \leq 10\hat{C}$ on $\widetilde{B}_0(\bar{x}, A) \times [-t_{\max} - \frac{1}{50}\eta^{-1}\hat{C}^{-1}, 0]$. So we can extract a smooth limit from the sequence to get the limiting solution which is defined on a larger time interval $[-t_{\max} - \frac{1}{50}\eta^{-1}\hat{C}^{-1}, 0]$. This contradicts the definition of the maximal time $-t_{\max}$.

It now remains to exclude possibility (1).

By using Li-Yau-Hamilton inequality [**18**] again, we have

$$\widetilde{R}_\infty(x, t) \leq \frac{t_{\max}}{t + t_{\max}}\widetilde{Q}.$$

So we only need to control the curvature near $-t_{\max}$. Exactly as in Step 4 of the proof of Theorem 4.1, it follows from Li-Yau-Hamilton inequality that

$$(5.19) \qquad d_0(x, y) \leq d_t(x, y) \leq d_0(x, y) + 30 t_{\max}\sqrt{\widetilde{Q}}$$

for any $x, y \in M_\infty$ and $t \in (-t_{\max}, 0]$.

Since the infimum of the scalar curvature is nondecreasing in time, we have some point $y_\infty \in M_\infty$ and some time $-t_{\max} < t_\infty < -t_{\max} + \frac{1}{50}\eta^{-1}$ such that $\widetilde{R}_\infty(y_\infty, t_\infty) < 5/4$. By (5.19), there is a constant $\widetilde{A} > 0$ such that $d_t(\bar{x}, y_\infty) \leq \widetilde{A}/2$ for all $t \in (-t_{\max}, 0]$.

Now we return back to the rescaled solution $\widetilde{g}_{ij}^{\alpha_m\beta_m}$. Clearly, for arbitrarily given small $\epsilon' > 0$, as m large enough, there is a point y_m in the underlying manifold of $\widetilde{g}_{ij}^{\alpha_m\beta_m}$ at time 0 satisfying the following properties

$$(5.20) \qquad \widetilde{R}(y_m, t_\infty) < \frac{3}{2}, \quad \widetilde{d}_t(\bar{x}, y_m) \leq \widetilde{A}$$

for $t \in [-t_{\max} + \epsilon', 0]$. By the definition of convergence, we know that for any fixed $A \geq 2\widetilde{A}$, as m large enough, the rescaled solution over $\widetilde{B}_0(\bar{x}, A)$ is defined on the time interval $[t_\infty, 0]$ and satisfies

$$\widetilde{R}(x, t) \leq \frac{2t_{\max}}{t + t_{\max}}\widetilde{Q}$$

on $\widetilde{B}_0(\bar{x}, A) \times [t_\infty, 0]$. Then by Assertion 2 of Step 2, we have proved there is a sufficiently large $\bar{m}_0$ such that as $m \geq \bar{m}_0$, the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m\beta_m}$ at y_m can be defined on $[-t_{\max} - \frac{1}{50}\eta^{-1}, 0]$, and satisfy

$$\widetilde{R}(y_m, t) \leq 15$$

for $t \in [-t_{\max} - \frac{1}{50}\eta^{-1}, t_\infty]$.

We now prove a statement analogous to Assertion 4 (i) of Step 3.

Assertion 5. For the above rescaled solutions $\widetilde{g}_{ij}^{\alpha_m\beta_m}$ and $\bar{m}_0$, we have that for any $L > 0$, there is a positive constant $\omega(L)$ such that the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m\beta_m}$ satisfy

$$\widetilde{R}(x,t) \leq \omega(L)$$

for all (x,t) with $\tilde{d}_t(x,y_m) \leq L$ and $t \in [-t_{\max} - \frac{1}{50}\eta^{-1}, t_\infty]$ and for all $m \geq \bar{m}_0$.

Proof of Assertion 5. We slightly modify the argument in the proof of Assertion 4 (i). Let

$$M(\rho) = \sup\{\widetilde{R}(x,t) \quad | \quad \tilde{d}_t(x,y_m) \leq \rho \text{ and } t \in [-t_{\max} - \frac{1}{50}\eta^{-1}, t_\infty]$$
$$\text{in the rescaled solutions } \widetilde{g}_{ij}^{\alpha_m\beta_m}, m \geq \bar{m}_0\}$$

and

$$\rho_0 = \sup\{\rho > 0 \mid M(\rho) < +\infty\}.$$

Note that the estimate (5.17) implies that $\rho_0 > 0$. We only need to show $\rho_0 = +\infty$.

We argue by contradiction. Suppose $\rho_0 < +\infty$. Then, after passing to subsequence, there are a sequence of $(\tilde{y}_m, t_m)$ in the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m\beta_m}$ with $t_m \in [-t_{\max} - \frac{1}{50}\eta^{-1}, t_\infty]$ and $\tilde{d}_{t_m}(y_m, \tilde{y}_m) \to \rho_0 < +\infty$ such that $\widetilde{R}(\tilde{y}_m, t_m) \to +\infty$. Denote by γ_m a minimizing geodesic segment from y_m to $\tilde{y}_m$ at the time t_m and denote by $\widetilde{B}_{t_m}(y_m, \rho_0)$ the geodesic open ball centered at y_m of radius ρ_0 on the rescaled solution $\widetilde{g}_{ij}^{\alpha_m\beta_m}(\cdot, t_m)$.

For any $0 < \rho < \rho_0$ with ρ near ρ_0, by applying Assertion 3 as before, we get that the rescaled solutions on the balls $\widetilde{B}_{t_m}(y_m, \rho)$ are defined on the time interval $[t_m - \frac{1}{16}\eta^{-1}M(\rho)^{-1}, t_m]$ for all large m. And by Step 1 and Shi's derivative estimate, we further know that the covariant derivatives of the curvatures of all order on $\widetilde{B}_{t_m}(y_m, \rho - \frac{(\rho_0-\rho)}{2}) \times [t_m - \frac{1}{32}\eta^{-1}M(\rho)^{-1}, t_m]$ are also uniformly bounded. Then by the uniform κ-noncollapsing and Hamilton's compactness theorem, after passing to a subsequence, we can assume that the marked sequence $(\widetilde{B}_{t_m}(y_m, \rho_0), \widetilde{g}_{ij}^{\alpha_m\beta_m}(\cdot, t_m), y_m)$ converges in C_{loc}^∞ topology to a marked (noncomplete) manifold $(B_\infty, \widetilde{g}_{ij}^\infty, y_\infty)$ and the geodesic segments γ_m converge to a geodesic segment (missing an endpoint) $\gamma_\infty \subset B_\infty$ emanating from y_∞.

Clearly, the limit also has restrictive isotropic curvature pinching (2.4). Then by repeating the same argument as in the proof of Assertion 4 (i) in the rest, we derive a contradiction with Hamilton's strong maximum principle. This proves Assertion 5.   q.e.d.

We then apply the second estimate of (5.20) and Assertion 5 to conclude that for any large constant $0 < A < +\infty$, there is a positive

constant $C(A)$ such that for any small $\epsilon' > 0$, the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m \beta_m}$ satisfy

$$(5.21) \qquad\qquad\qquad \widetilde{R}(x,t) \le C(A),$$

for all $x \in \widetilde{B}_0(\bar{x}, A)$ and $t \in [-t_{\max} + \epsilon', 0]$, and for all sufficiently large m. Then by applying Assertion 1 in Step 2, we conclude that the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m \beta_m}$ on the geodesic balls $\widetilde{B}_0(\bar{x}, A)$ are also defined on the extended time interval $[-t_{\max} + \epsilon' - \frac{1}{8}\eta^{-1}C(A)^{-1}, 0]$ for all sufficiently large m. Furthermore, by the gradient estimates as in Step 1, we have

$$\widetilde{R}(x,t) \le 10C(A),$$

for $x \in \widetilde{B}_0(\bar{x}, A)$ and $t \in [-t_{\max} + \epsilon' - \frac{1}{8}\eta^{-1}C(A)^{-1}, 0]$. Since $\epsilon' > 0$ is arbitrarily small, the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m \beta_m}$ on $\widetilde{B}_0(\bar{x}, A)$ are defined on the extended time interval $[-t_{\max} - \frac{1}{16}\eta^{-1}C(A)^{-1}, 0]$ and satisfy

$$(5.22) \qquad\qquad\qquad \widetilde{R}(x,t) \le 10C(A),$$

for $x \in \widetilde{B}_0(\bar{x}, A)$ and $t \in [-t_{\max} - \frac{1}{16}\eta^{-1}C(A)^{-1}, 0]$, and for all sufficiently large m.

Now, by taking convergent subsequences from the rescaled solutions $\widetilde{g}_{ij}^{\alpha_m \beta_m}$, we see that the limit solution is defined smoothly on a space-time open subset of $M_\infty \times (-\infty, 0]$ containing $M_\infty \times [-t_{\max}, 0]$. By Step 4, we see that the limiting metric $\widetilde{g}_{ij}^\infty(\cdot, -t_{\max})$ at time $-t_{\max}$ has bounded curvature. Then by combining with the 2ε-canonical neighborhood assumption we conclude that the curvature of the limit is uniformly bounded on the time interval $[-t_{\max}, 0]$. So we have excluded possibility (1).

Hence we have proved a subsequence of the rescaled solutions converges to an ancient κ-solution.

Finally, by combining with the canonical neighborhood theorem of ancient κ-solutions with restricted isotropic curvature pinching condition (Theorem 3.8) and the same argument in the second paragraph of Section 4, we see that $(\bar{x}, \bar{t})$ has a canonical neighborhood with parameter ε, which is a contradiction. Therefore we have completed the proof of the proposition.   q.e.d.

Summing up, we have proved that for an arbitrarily given compact four-manifold with positive isotropic curvature and with no essential incompressible space form, there exist non-increasing positive (continuous) functions $\widetilde{\delta}(t)$ and $\widetilde{r}(t)$, defined on $[0, +\infty)$, such that for an arbitrarily given positive (continuous) function $\delta(t)$ with $\delta(t) < \widetilde{\delta}(t)$ on $[0, +\infty)$, the Ricci flow with surgery, with the given four-manifold as initial datum, has a solution on a maximal time interval $[0, T)$, with

$T \leq 2/R_{min}(0) < +\infty$, obtained by evolving the Ricci flow and by performing δ-cutoff surgeries at a sequence of times $0 < t_1 < t_2 < \cdots < t_i < \cdots < T$ with $\delta(t_i) \leq \delta \leq \widetilde{\delta}(t_i)$ at each time t_i, so that the pinching assumption and the canonical neighborhood assumption with $r = \widetilde{r}(t)$ are satisfied. (At this moment we still do not know whether the surgery times t_i are discrete).

Clearly, the upper derivative of the volume in time satisfies

$$\frac{d}{dt}V(t) \leq 0$$

since the scalar curvature is nonnegative. Thus

$$V(t) \leq V(0)$$

for all $t \in [0, T)$. Also note that at each time t_i, the volume which is cut down by $\delta(t_i)$-cutoff surgery is at least an amount of $h^4(t_i)$ with $h(t_i)$ depending only on $\delta(t_i)$ and $\widetilde{r}(t_i)$ (by Lemma 5.2). Thus the set of the surgery times $\{t_i\}$ must be finite. So we have proved the following long-time existence result.

Theorem 5.6. *Given a compact four-dimensional Riemannian manifold with positive isotropic curvature and with no essential incompressible space form, and given any fixed small constant $\varepsilon > 0$, there exist non-increasing positive (continuous) functions $\widetilde{\delta}(t)$ and $\widetilde{r}(t)$, defined on $[0, +\infty)$, such that for arbitrarily given positive (continuous) function $\delta(t)$ with $\delta(t) \leq \widetilde{\delta}(t)$ on $[0, +\infty)$, the Ricci flow with surgery, with the given four-manifold as initial datum, has a solution satisfying the the pinching assumption and the canonical neighborhood assumption (with accuracy ε) with $r = \widetilde{r}(t)$ on a maximal time interval $[0, T)$ with $T < +\infty$ and becoming extinct at T, which is obtained by evolving the Ricci flow and by performing a finite number of cutoff surgeries with each δ-cutoff at time $t \in (0, T)$ having $\delta = \delta(t)$. Consequently, the initial manifold is diffeomorphic to a connected sum of a finite copies of $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{S}^1$, and $\mathbb{S}^3 \widetilde{\times} \mathbb{S}^1$.*

Finally, the main theorem (Theorem 1.1) stated in Section 1 is a direct consequence of the above theorem.   q.e.d.

Appendix A. Standard Solutions

In this appendix, we will prove the curvature estimates for the standard solutions, and give a canonical neighborhood description for the standard solution in dimension four. We have used these estimates and the description in Section 5 for the surgery arguments. The curvature estimate for the special case that the dimension is three and the initial metric is rotationally symmetric, was earlier claimed by Perelman in [**32**].

Theorem A.1. *Let g_{ij} be a complete Riemannian metric on $\mathbb{R}^n$ ($n > 2$) with nonnegative curvature operator and with positive scalar curvature which is asymptotic to a round cylinder of scalar curvature 1 at infinity. Then there is a complete solution $g_{ij}(\cdot, t)$ to the Ricci flow, with g_{ij} as initial metric, which exists on the time interval $[0, \frac{n-1}{2})$, has bounded curvature in each closed time interval $[0, t] \subset [0, \frac{n-1}{2})$, and satisfies the estimate*

$$R(x, t) \geq \frac{C^{-1}}{\frac{n-1}{2} - t}$$

for some C depending only on the initial metric g_{ij}.

Proof. Since the initial metric has bounded curvature operator and has a positive lower bound on its scalar curvature, by [**36**] and the maximum principle, the Ricci flow has a solution $g(\cdot, t)$ on a maximal time interval $[0, T)$ with $T < \infty$. By Hamilton's maximum principle, the solution $g(x, t)$ has nonnegative curvature operator for $t > 0$. Note that the injectivity radius of the initial metric has a positive lower bound, so by the same proof of Perelman's no local collapsing theorem I (in Section 7.3 of [**31**], or see the proof of Theorem 3.5 of this paper), there is a $\kappa = \kappa(T, g_{ij}) > 0$ such that $g_{ij}(\cdot, t)$ is κ-noncollapsed on the scale $\sqrt{T}$.

We will firstly prove the following assertion.

Claim 1. There is a positive function $\omega : [0, \infty) \longrightarrow [0, \infty)$ depending only on the initial metric and κ such that

$$R(x, t) \leq R(y, t)\omega(R(y, t)d_t^2(x, y))$$

for all $x, y \in M^n = \mathbb{R}^n$, $t \in [0, T)$.

The proof is similar to that of Proposition 3.3. Notice that the initial metric has nonnegative curvature operator and its scalar caurvature satisfies

(A.1) $C^{-1} \leq R(x) \leq C$

for some positive constant $C > 1$. By maximum principle, we know $T \geq \frac{1}{2nC}$ and $R(x, t) \leq 2C$ for $t \in [0, \frac{1}{4nC}]$. The assertion is clearly true for $t \in [0, \frac{1}{4nC}]$.

Now fix $(y, t_0) \in M^n \times [0, T)$ with $t_0 \geq \frac{1}{4nC}$. Let z be the closest point to y with the property $R(z, t_0)d_{t_0}^2(z, y) = 1$ (at time t_0). Draw a shortest geodesic from y to z and choose a point $\tilde{z}$ on the geodesic satisfying $d_{t_0}(z, \tilde{z}) = \frac{1}{4}R(z, t_0)^{-\frac{1}{2}}$; then we have

$$R(x, t_0) \leq \frac{1}{(\frac{1}{2}R(z, t_0)^{-\frac{1}{2}})^2}, \qquad \text{on } B_{t_0}\left(\tilde{z}, \frac{1}{4}R(z, t_0)^{-\frac{1}{2}}\right).$$

Note that $R(x,t) \geq C^{-1}$ everywhere by the evolution equation of the scalar curvature. Then by Li-Yau-Hamilton inequality [18], for all $(x,t) \in B_{t_0}(\tilde{z}, \frac{1}{8nC}R(z,t_0)^{-\frac{1}{2}}) \times [t_0 - (\frac{1}{8nC}R(z,t_0)^{-\frac{1}{2}})^2, t_0]$, we have

$$R(x,t) \leq \left(\frac{t_0}{t_0 - (\frac{1}{8n\sqrt{C}})^2} \right) \frac{1}{(\frac{1}{2}R(z,t_0)^{-\frac{1}{2}})^2},$$

$$\leq \left[\frac{1}{8nC}R(z,t_0)^{-\frac{1}{2}} \right]^{-2}$$

Combining this with the κ-noncollapsing, we have

$$\text{Vol}\left(B_{t_0}\left(\tilde{z}, \frac{1}{8nC}R(z,t_0)^{-\frac{1}{2}} \right) \right) \geq \kappa \left(\frac{1}{8nC}R(z,t_0)^{-\frac{1}{2}} \right)^n$$

and then

$$\text{Vol}(B_{t_0}(z, 8R(z,t_0)^{-\frac{1}{2}})) \geq \kappa \left(\frac{1}{64nC} \right)^n (8R(z,t_0)^{-\frac{1}{2}})^n.$$

So by Corollary 11.6 (b) of [31], there hold

$$R(x,t_0) \leq C(\kappa)R(z,t_0), \quad \text{for all } x \in B_{t_0}(z, 2R(z,t_0)^{-\frac{1}{2}}).$$

Here in the following we denote by $C(\kappa)$ various positive constants depending only on κ, n and the initial metric.

Now by Li-Yau-Hamilton inequality [18] and local gradient estimate of Shi [36], we obtain

$$R(x,t) \leq C(\kappa)R(z,t_0), \quad \text{and} \quad \left| \frac{\partial}{\partial t}R \right|(x,t) \leq C(\kappa)(R(z,t_0))^2$$

for all $(x,t) \in B_{t_0}(z, 2R(z,t_0)^{-\frac{1}{2}}) \times [t_0 - (\frac{1}{8nC}R(z,t_0)^{-\frac{1}{2}})^2, t_0]$. Therefore by combining with the Harnack estimate [18], we obtain

$$R(y,t_0) \geq C(\kappa)^{-1}R(z, t_0 - C(\kappa)^{-1}R(z,t_0)^{-1})$$

$$\geq C(\kappa)^{-2}R(z,t_0).$$

Consequently, we have showed that there is a constant $C(\kappa)$ such that

$$\text{Vol}(B_{t_0}(y, R(y,t_0)^{-\frac{1}{2}})) \geq C(\kappa)^{-1}(R(y,t_0)^{-\frac{1}{2}})^n$$

and

$$R(x,t_0) \leq C(\kappa)R(y,t_0) \quad \text{for all} \quad x \in B_{t_0}(y, R(y,t_0)^{-\frac{1}{2}}).$$

In general, for any $r \geq R(y,t_0)^{-\frac{1}{2}}$, we have

$$\text{Vol}(B_{t_0}(y,r)) \geq C(\kappa)^{-1}(r^2 R(y,t_0))^{-\frac{n}{2}}r^n.$$

By applying Corollary 11.6 of [31] again, there exists a positive constant $\omega(r^2 R(y,t_0))$ depending only on the constant $r^2 R(y,t_0)$ and κ such that

$$R(x,t_0) \leq R(y,t_0)\omega(r^2 R(y,t_0)), \quad \text{for all } x \in B_{t_0}\left(y, \frac{1}{4}r \right).$$

This proves the desired Claim 1.

Now we study the asymptotic behavior of the solution at infinity. For any $0 < t_0 < T$, we know that the metrics $g_{ij}(x,t)$ with $t \in [0, t_0]$ have uniformly bounded curvature by the definition of T. Let x_k be a sequence of points with $d_0(x_0, x_k) \mapsto \infty$. By Hamilton's compactness theorem [19], after taking a subsequence, $g_{ij}(x,t)$ around x_k will converge to a solution to the Ricci flow on $\mathbb{R} \times \mathbb{S}^{n-1}$ with round cylinder metric of scalar curvature 1 as initial data. Denote the limit by $\tilde{g}_{ij}$. Then by the uniqueness theorem in [11], we have

$$\tilde{R}(x,t) = \frac{\frac{n-1}{2}}{\frac{n-1}{2} - t}, \quad \text{for all} \quad t \in [0, t_0].$$

It follows that $T \leq \frac{n-1}{2}$. In order to show $T = \frac{n-1}{2}$, it suffices to prove the following assertion:

Claim 2. Suppose $T < \frac{n-1}{2}$. Fix a point $x_0 \in M^n$, then there is a $\delta > 0$, such that for any $x \in M$ with $d_0(x, x_0) \geq \delta^{-1}$, we have

$$R(x,t) \leq 2C + \frac{n-1}{\frac{n-1}{2} - t} \quad \text{for all} \quad t \in [0, T)$$

where C is the constant in (A.1).

In view of Claim 1, if Claim 2 holds, then

$$\sup_{M^n \times [0,T)} R(y,t) \leq \omega \left(\delta^{-2} \left(2C + \frac{n-1}{\frac{n-1}{2} - T} \right) \right) \left(2C + \frac{n-1}{\frac{n-1}{2} - T} \right)$$
$$< \infty,$$

which will contradict with the definition of T.

To show Claim 2, we argue by contradiction. Suppose for each $\delta > 0$, there is a (x_δ, t_δ) with $0 < t_\delta < T$ such that

$$R(x_\delta, t_\delta) > 2C + \frac{n-1}{\frac{n-1}{2} - t_\delta} \quad \text{and} \quad d_0(x_\delta, x_0) \geq \delta^{-1}.$$

Let

$$\bar{t}_\delta = \sup \left\{ t \, \Big| \, \sup_{M^n \setminus B_0(x_0, \delta^{-1})} R(y,t) < 2C + \frac{n-1}{\frac{n-1}{2} - t} \right\}.$$

Since $\lim\limits_{d_0(y,x_0) \to \infty} R(y,t) = \frac{n-1}{2} / (\frac{n-1}{2} - t)$ and $\sup_{M \times [0, \frac{1}{4nC}]} R(y,t) \leq 2C$, we know $\frac{1}{4nC} \leq \bar{t}_\delta \leq t_\delta$ and there is a $\bar{x}_\delta$ such that $d_0(x_0, \bar{x}_\delta) \geq \delta^{-1}$ and $R(\bar{x}_\delta, \bar{t}_\delta) = 2C + n - 1 / (\frac{n-1}{2} - \bar{t}_\delta)$. By Claim 1 and Hamilton's compactness theorem [19], as $\delta \to 0$ and after taking subsequence, the metrics $g_{ij}(x,t)$ on $B_0(\bar{x}_\delta, \frac{\delta^{-1}}{2})$ over the time interval $[0, \bar{t}_\delta]$ will converge to a solution $\tilde{g}$ on $\tilde{M} = \mathbb{R} \times \mathbb{S}^{n-1}$ with standard metric of scalar curvature

1 as initial datum over the time interval $[0, \bar{t}_\infty]$, and its scalar curvature satisfies

$$
\begin{aligned}
\tilde{R}(\bar{x}_\infty, \bar{t}_\infty) &= 2C + \frac{n-1}{\frac{n-1}{2} - \bar{t}_\infty}, \\
\tilde{R}(x,t) &\leq 2C + \frac{n-1}{\frac{n-1}{2} - \bar{t}_\infty}, \quad \text{for all } t \in [0, \bar{t}_\infty],
\end{aligned}
$$

where $(\bar{x}_\infty, \bar{t}_\infty)$ is the limit of $(\bar{x}_\delta, \bar{t}_\delta)$. On the other hand, by the uniqueness theorem in [11] again, we know

$$
\tilde{R}(\bar{x}_\infty, \bar{t}_\infty) = \frac{\frac{n-1}{2}}{\frac{n-1}{2} - \bar{t}_\infty},
$$

which is a contradiction. Hence we have proved Claim 2 and then have verified $T = \frac{n-1}{2}$.

Now we are ready to show

$$
(A.2) \qquad R(x,t) \geq \frac{\tilde{C}^{-1}}{\frac{n-1}{2} - t}, \quad \text{for all } (x,t) \in M^n \times [0, \frac{n-1}{2}),
$$

for some positive constant $\tilde{C}$ depending only on the initial metric.

For any $(x,t) \in M^n \times [0, \frac{n-1}{2})$, by Claim 1 and κ-noncollapsing, there is a constant $C(\kappa) > 0$ such that

$$
\text{Vol}_t(B_t(x, R(x,t)^{-\frac{1}{2}})) \geq C(\kappa)^{-1}(R(x,t)^{-\frac{1}{2}})^n.
$$

Then by the volume estimate of Calabi-Yau [34] on manifolds with nonnegative Ricci curvature, for any $a \geq 1$, we have

$$
\text{Vol}_t(B_t(x, aR(x,t)^{-\frac{1}{2}})) \geq C(\kappa)^{-1} \frac{a}{8n}(R(x,t)^{-\frac{1}{2}})^n.
$$

On the other hand, since $(M^n, g_{ij}(\cdot, t))$ is asymptotic to a cylinder of scalar curvature $\frac{n-1}{2}/(\frac{n-1}{2} - t)$, for sufficiently large $a > 0$, we have

$$
\text{Vol}_t\left(B_t\left(x, a\sqrt{\frac{n-1}{2} - t}\right)\right) \leq C(n)a\left(\frac{n-1}{2} - t\right)^{\frac{n}{2}}.
$$

Combining these two inequalities, we have for all sufficiently large a:

$$
\begin{aligned}
C(n)a\left(\frac{n-1}{2} - t\right)^{\frac{n}{2}} &\geq \text{Vol}_t\left(B_t\left(x, a\left(\frac{\sqrt{\frac{n-1}{2} - t}}{R(x,t)^{-\frac{1}{2}}}\right)R(x,t)^{-\frac{1}{2}}\right)\right) \\
&\geq C(\kappa)^{-1}\frac{a}{8n}\left(\frac{\sqrt{\frac{n-1}{2} - t}}{R(x,t)^{-\frac{1}{2}}}\right)(R(x,t)^{-\frac{1}{2}})^n,
\end{aligned}
$$

which gives the desired estimate (A.2). Therefore we complete the proof of the theorem.   q.e.d.

We now fix a standard capped infinite cylinder metric on $\mathbb{R}^4$ as follows. Consider the semi-infinite standard round cylinder $N_0 = \mathbb{S}^3 \times (-\infty, 4)$ with the metric g_0 of scalar curvature 1. Denote by z the coordinate of the second factor $(-\infty, 4)$. Let f be a smooth nondecreasing convex function on $(-\infty, 4)$ defined by

$$\begin{cases} f(z) = 0, & z \le 0, \\[2mm] f(z) = ce^{-\frac{D}{z}}, & z \in (0, 3], \\[2mm] f(z) \text{ is strictly convex on } z \in [3, 3.9], \\[2mm] f(z) = -\frac{1}{2}\log(16 - z^2), & z \in [3.9, 4), \end{cases}$$

where the small (positive) constant $c = c_0$ and big (positive) constant $D = D_0$ are fixed as in Lemma 5.3. Let us replace the standard metric g_0 on the portion $\mathbb{S}^3 \times [0, 4)$ of the semi-infinite cylinder by $\hat{g} = e^{-2f}g_0$. Then the resulting metric $\hat{g}$ will be smooth on $\mathbb{R}^4$ obtained by adding a point to $\mathbb{S}^3 \times (-\infty, 4)$ at $z = 4$. We denote the manifold by $(\mathbb{R}^4, \hat{g})$.

Next we will consider the "canonical neighborhood" decomposition of the fixed standard solution with $(\mathbb{R}^4, \hat{g})$ as initial metric.

Corollary A.2. *Let $g_{ij}(x, t)$ be the above fixed standard solution to the Ricci flow on $\mathbb{R}^4 \times [0, \frac{3}{2})$. Then for any $\varepsilon > 0$, there is a positive constant $C(\varepsilon)$ such that each point $(x, t) \in \mathbb{R}^4 \times [0, \frac{3}{2})$ has an open neighborhood B, with $B_t(x, r) \subset B \subset B_t(x, 2r)$ for some $0 < r < C(\varepsilon)R(x, t)^{-\frac{1}{2}}$, which falls into one of the following two categories: either*

(a) *B is an ε-cap, or*
(b) *B is an ε-neck and it is the slice at the time t of the parabolic neighborhood $P(x, t, \varepsilon^{-1}R(x, t)^{-\frac{1}{2}}, -\min\{R(x, t)^{-1}, t\})$, on which the standard solution is, after scaling with the factor $R(x, t)$ and shifting the time t to zero, ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the corresponding subset of the evolving standard cylinder $\mathbb{S}^3 \times \mathbb{R}$ over the time interval $[-\min\{tR(x, t), 1\}, 0]$ with scalar curvature 1 at the time zero.*

Proof. First, we discuss the curvature pinching of this fixed standard solution. Because the initial metric is asymptotic to a cylinder, we have a uniform isotropic curvature pinching at initial, that is to say, there is a universal constant $\Lambda' > 0$ such that

$$\max\{a_3, b_3, c_3\} \le \Lambda' a_1 \text{ and } \max\{a_3, b_3, c_3\} \le \Lambda' c_1.$$

Moreover since the initial metric has nonnegative curvature operator, we have $b_3^2 \le a_1 c_1$. By the pinching estimates of Hamilton [**16**] [**21**], $b_3^2 \le a_1 c_1$ is preserved, and the following two estimates are also preserved

$$\max\{a_3, b_3, c_3\} \le \max\{\Lambda', 5\}a_1 \text{ and } \max\{a_3, b_3, c_3\} \le \max\{\Lambda', 5\}c_1,$$

under the Ricci flow.

The proof of the lemma is reduced to two assertions. We now state and prove the first assertion, which takes care of those points with times close to $\frac{3}{2}$.

Assertion 1. For any $\varepsilon > 0$, there is a positive number $\theta = \theta(\varepsilon)$ with $0 < \theta < \frac{3}{2}$ such that for any $(x_0, t_0) \in \mathbb{R}^4 \times [\theta, \frac{3}{2})$, the standard solution on the parabolic neighborhood

$$P(x_0, t_0, \varepsilon^{-1} R(x_0, t_0)^{-\frac{1}{2}}, -\varepsilon^{-2} R(x_0, t_0)^{-1})$$

is well-defined and is, after scaling with the factor $R(x_0, t_0)$, ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the corresponding subset of some oriented ancient-κ solution with restricted isotropic curvature pinching (2.4).

We argue by contradiction. Suppose Assertion 1 is not true, then there exists $\varepsilon_0 > 0$ and a sequence of points (x_k, t_k) with $t_k \to \frac{3}{2}$, such that the standard solution on the parabolic neighborhoods

$$P(x_k, t_k, \varepsilon_0^{-1} R(x_k, t_k)^{-\frac{1}{2}}, -\varepsilon_0^{-2} R(x_k, t_k)^{-1})$$

is not, after scaling by the factor $R(x_k, t_k)$, ε_0-close to the corresponding subset of any ancient κ-solution. Note that by Theorem A.1, there is a constant $C > 0$ (depending only on the initial metric, hence it is universal) such that $R(x, t) \geq C^{-1}/(\frac{3}{2} - t)$. This implies

$$\varepsilon_0^{-2} R(x_k, t_k)^{-1} \leq C\varepsilon_0^{-2} \left(\frac{3}{2} - t_k\right) < t_k,$$

and then the standard solution on the parabolic neighborhoods

$$P(x_k, t_k, \varepsilon_0^{-1} R(x_k, t_k)^{-\frac{1}{2}}, -\varepsilon_0^{-2} R(x_k, t_k)^{-1})$$

is well-defined as k large. By Claim 1 in Theorem A.1, there is a positive function $\omega : [0, \infty) \to [0, \infty)$ such that

$$R(x, t_k) \leq R(x_k, t_k)\omega(R(x_k, t_k)d_{t_k}^2(x, x_k))$$

for all $x \in \mathbb{R}^4$. Now by scaling the standard solution $g_{ij}(\cdot, t)$ around (x_k, t_k) with the factor $R(x_k, t_k)$ and shifting the time t_k to zero, we get a sequence of the rescaled solutions to the Ricci flow $\tilde{g}_{ij}^k(x, \tilde{t}) = R(x_k, t_k)g_{ij}(x, t_k + \tilde{t}/R(x_k, t_k))$ defined on $\mathbb{R}^4$ with $\tilde{t} \in [-R(x_k, t_k)t_k, 0]$. We denote the scalar curvature and the distance of the rescaled metric $\tilde{g}_{ij}^k$ by $\tilde{R}^k$ and $\tilde{d}$. By combining with Claim 1 in Theorem A.1 and the Li-Yau-Hamilton inequality, we get

$$\tilde{R}^k(x, 0) \leq \omega(\tilde{d}_0^2(x, x_k))$$
$$\tilde{R}^k(x, \tilde{t}) \leq \frac{R(x_k, t_k)t_k}{\tilde{t} + R(x_k, t_k)t_k}\omega(\tilde{d}_0^2(x, x_k))$$

for any $x \in \mathbb{R}^4$ and $\tilde{t} \in (-R(x_k, t_k)t_k, 0]$. Note that $R(x_k, t_k)t_k \to \infty$ by Theorem A.1. We have shown in the proof of Theorem A.1 that the

standard solution is κ-noncollapsed on all scales less than 1 for some $\kappa > 0$. Then from the κ-noncollapsing, the above curvature estimates, and Hamilton's compactness theorem (Theorem 16.1 of [20]), we know $\tilde{g}_{ij}^k(x, \tilde{t})$ has a convergent subsequence (as $k \to \infty$) whose limit is an ancient, κ-noncollapsed, complete and oriented solution with nonnegative curvature operator. This limit must have bounded curvature by the same proof of Step 3 in Theorem 4.1. It also satisfies the restricted isotropic pinching condition (2.4). This gives a contradiction. Assertion 1 is proved.

We now fix the constant $\theta(\varepsilon)$ obtained in Assertion 1. Let O be the tip of the manifold $\mathbb{R}^4$ (it is rotationally symmetric about O at time 0; it remains so as $t > 0$ by the uniqueness Theorem [11]).

Assertion 2. There are constants $B_1(\varepsilon)$, $B_2(\varepsilon)$ depending only on ε, such that if $(x_0, t_0) \in M \times [0, \theta)$ with $d_{t_0}(x_0, O) \leq B_1(\varepsilon)$, then there is a $0 < r < B_2(\varepsilon)$ such that $B_{t_0}(x_0, r)$ is an ε-cap; if $(x_0, t_0) \in M \times [0, \theta)$ with $d_{t_0}(x_0, O) \geq B_1(\varepsilon)$, then the parabolic neighborhood

$$P\left(x_0, t_0, \varepsilon^{-1} R(x_0, t_0)^{-\frac{1}{2}}, -\min\{R(x_0, t_0)^{-1}, t_0\}\right)$$

is, after scaling with the factor $R(x_0, t_0)$ and shifting the time t_0 to zero, ε-close (in $C^{[\varepsilon^{-1}]}$ topology) to the corresponding subset of the evolving standard cylinder $\mathbb{S}^3 \times \mathbb{R}$ over the time interval $[-\min\{t_0 R(x_0, t_0), 1\}, 0]$ with scalar curvature 1 at the time zero.

Since the standard solution exists on the time interval $[0, \frac{3}{2})$, there is a constant $B_0(\varepsilon)$ such that the curvatures on $[0, \theta(\varepsilon)]$ are uniformly bounded by $B_0(\varepsilon)$. This implies that the metrics in $[0, \theta(\varepsilon)]$ are equivalent. Note that the initial metric is asymptotic to a standard cylinder. For any sequence of points x_k with $d_0(O, x_k) \to \infty$, after taking a subsequence, $g_{ij}(x, t)$ around x_k will converge to a solution to the Ricci flow on $\mathbb{R} \times \mathbb{S}^3$ with round cylinder metric of scalar curvature 1 as initial data. By the uniqueness theorem [11], the limit solution must be the standard evolving round cylinder. This implies that there is a constant $B_1(\varepsilon) > 0$ depending on ε such that for any (x_0, t_0) with $t_0 \leq \theta(\varepsilon)$ and $d_{t_0}(x, O) \geq B_1(\varepsilon)$, the standard solution on the parabolic neighborhood $P(x_0, t_0, \varepsilon^{-1} R(x_0, t_0)^{-\frac{1}{2}}, -\min\{R(x_0, t_0)^{-1}, t_0\})$ is, after scaling with the factor $R(x_0, t_0)$, ε-close to the corresponding subset of the evolving round cylinder. Since the solution is rotationally symmetric around O, the cap neighborhood structures of those points x_0 with $d_{t_0}(x_0, O) \leq B_1(\varepsilon)$ follows directly. Assertion 2 is proved.

Therefore we finish the proof of Corollary A.2.   q.e.d.

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